Chapter 1: Approximation and Fourier Analysis

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Local Smoothness Conditions on a Function Which Guarantee Convergence of Double Walsh-Fourier Series of This Function

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Abstract. The local smoothness conditions on a function are obtained, which guarantee convergence almost everywhere on some set of positive measure of the double Walsh-Fourier series of this function summed over rectangles.

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1. Discussion and Setting of the Problem

Studies on convergence (including convergence almost everywhere) of series with respect to the classical orthonormal systems (in particular, the trigonometric and the Walsh systems) is one of the central problems in the modern theory of Fourier series.

In the present paper we shall consider Fourier series with respect to the Walsh-Paley system (which have different applications and, in particular, are used in the digital data processing).

As it is known, in 1961 E.Stein [1] proved that the one-dimensional Walsh-Fourier series of a function $f \in L_1(\mathbb{I}^1)$, where $\mathbb{I}^1 = [0,1)$, can unboundedly diverge almost everywhere (a.e.) on $\mathbb{I}^1$. Moreover, in 2004 S.V.Bochkarev [2] obtained the following result: there exists a function $f \in \Phi(\mathbb{I}^1) = \Phi_L(\mathbb{I}^1)$ (where $\Phi_u = u\varphi(u)$, and $\varphi(u)$ is a non-decreasing on $[0,\infty)$ function, $\varphi(0) = 1$ and $\varphi(u) = o((\log u)^{\frac{1}{2}})$ as $u \to \infty$), whose Walsh-Fourier series unboundedly diverges everywhere on $\mathbb{I}^1$. On the other hand, as it was proved in 2003 by P.Sjolin and F.Soria [3], if a

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function $f \in \mathcal{F}_1(\mathbb{I}^1) = L(\log^+ L)(\log^+ \log^+ \log^+ L)(\mathbb{I}^1)$ then Walsh-Fourier series of this function already converges a.e. on $\mathbb{I}^1$.

The question arises: if for some measurable set $E \subset \mathbb{I}^1$, $\mu E > 0$ (μ is the Lebesgue measure on line) a function $f \in \mathcal{F}_1(E) \cap \Phi(\mathbb{I}^1)$, or (in the "scale" of Lebesgue classes) a function $f \in L_p(E) \cap L_1(\mathbb{I}^1)$, $p > 1$, then what can be said about convergence a.e. of the one-dimensional Walsh-Fourier series of this function, in particular, about convergence a.e. on the set E (where the function f is "sufficiently smooth") or on some of its subsets $E_1 \subset E$, $\mu E_1 > 0$?

In this case, the following question naturally arises: what must be the structure of the set E – open, closed, G_δ, etc., what must be its boundary.

The analogous question can be posed as well for the N - dimensional ($N > 1$) Walsh-Fourier series, namely: on what measurable subsets $E \subset \mathbb{I}^N$, where $\mathbb{I}^N = [0,1)^N$ is the N-dimensional cube, it is possible to "localize" these or those conditions on a function f, defined on the whole $\mathbb{I}^N$, which "guarantee" convergence a.e. on the whole $\mathbb{I}^N$ of the multiple Walsh-Fourier series summed over rectangles. In the multiple case besides the question concerning the structural characteristics of the set E the question arises concerning the geometric characteristics of this set.

Denote as $\mathcal{F}(\mathbb{I}^N)$ the class of summable (on $\mathbb{I}^N$) functions such that for any f in this class ($f \in \mathcal{F}(\mathbb{I}^N)$) the multiple Walsh-Fourier series (summed over rectangles) of the function f converges a.e. on $\mathbb{I}^N$. So, we are interested in the question concerning correlation between the *structural and geometric characteristics of the set* E and the *smoothness of the function* in the framework of these or those subspaces $\mathcal{F}$ of the space L_1.

In the present paper we shall give some solutions of the posed question for double Walsh-Fourier series summed over rectangles.

As to the one-dimensional case, taking account of the classical principle of localization (see [4] or [5, p. 70])[1], and the mentioned earlier result by P.Sjolin and F.Soria [3], we can give a partial answer to the posed above question: for any open a.e.[2] set E, $E \subset \mathbb{I}^1$, $\mu E > 0$ and for any function $f \in \mathcal{F}_1(E) \cap \Phi(\mathbb{I}^1)$ (for any function $f \in L_p(E) \cap L_1(\mathbb{I}^1)$, $p > 1$) the one-dimensional Walsh-Fourier series of this function converges a.e. on the set E.

Note that in the setting of the problem we posed the question about convergence a.e. of Walsh-Fourier series in the classes $\mathcal{F}(E) \cap L_1(\mathbb{I}^1)$ on the set E (or on some of its subsets $E_1 \subset E$), and this is connected with the fact that outside the set E the Walsh-Fourier series of a function $f \in \mathcal{F}(E) \cap L_1(\mathbb{I}^1)$ can, in general, diverge. For example, it is not difficult to prove (taking account of [2] and [4]), that for any open set E with boundary of measure zero or for any closed set E, $E \subset \mathbb{I}^1$, $\mu E > 0$ there exists a function $f \in L_\infty(E) \cap \Phi(\mathbb{I}^1)$, whose Walsh-Fourier series unboundedly diverges a.e. outside the set E.

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[1] Walsh-Fourier series of a function $f \in L_1(\mathbb{I}^1), f(x) = 0$ on an open interval $J \subset \mathbb{I}^1$ converges uniformly to zero on each segment which is entirely contained in J.

[2] The set E is called *open a.e.*, if there exists an open set E_1 such that $\mu(E \triangle E_1) = 0$.

For trigonometric Fourier series investigations of this type were carried in the one-dimensional case by G.Alexits, N.K.Bari, S.B.Stechkin, P.L.Ul'yanov (see [6, p. 350-354]), and in the multi-dimensional case ($N > 1$) by I.L.Bloshanskii [7].

2. Notation

Let us denote as $\{\omega_n\}_{n=0}^{\infty} = \{\omega_n(x)\}_{n=0}^{\infty}$, $x \in [0,1) = \mathbb{I}^1$ the Walsh system in Paley enumeration (see, e.g., [5]), i.e. the system of functions constructed as follows. Let us consider the function

$$r_0(x) = \begin{cases} 1, & \text{for } x \in [0, \frac{1}{2}), \\ -1, & \text{for } x \in [\frac{1}{2}, 1). \end{cases}$$

Continue this function with period 1 to the entire number line and define the Rademacher system $\{r_k\}_{k=0}^{\infty}$ by setting $r_k(x) = r_0(2^k x)$, $k = 0, 1, \dots$.

Next, we represent each positive integer m as the sum $m = \sum_{i=0}^{k} \varepsilon_i 2^i$, with $\varepsilon_i = 0$ or 1 for $i = 0, 1, \dots, k-1$ and $\varepsilon_k = 1$.

The Walsh functions $\omega_m(x)$ are defined as follows: $\omega_0(x) \equiv 1$,

$$\omega_m(x) = \prod_{i=0}^{k} (r_i(x))^{\varepsilon_i}, \quad m = 1, 2, \dots .$$

Note that the system $\{\omega_n\}_{n=0}^{\infty}$ is orthonormal on $\mathbb{I}^1$ and complete in the space $L_p(\mathbb{I}^1)$ for each p, $1 \le p < \infty$.

Let $\mathbb{Z}^N$, $\mathbb{Z}^N \subset \mathbb{R}^N$, $N \ge 1$ be a set of all vectors with integer coordinates, assume $\mathbb{Z}_\alpha^N = \{n = (n_1, \dots, n_N) \in \mathbb{Z}^N : n_j \ge \alpha,\ j = 1, \dots, N\}$, $\alpha \in \mathbb{Z}^1$. Further, for $x = (x_1, \dots, x_N) \in \mathbb{I}^N$, where $\mathbb{I}^N = [0,1)^N$ and $k = (k_1, \dots, k_N) \in \mathbb{Z}_0^N$, denote as $\omega_k(x) = \omega_{k_1}(x_1) \times \cdots \times \omega_{k_N}(x_N)$ the multiple Walsh-Paley system. Let a function $f \in L_1(\mathbb{I}^N)$ be expanded into a multiple Walsh-Fourier series with respect to the system $\{\omega_k\}_{k \in \mathbb{Z}_0^N}$:

$$f(x) \sim \sum_{k \in \mathbb{Z}_0^N} c_k\, \omega_k(x),$$

where

$$c_k = c_{k_1,\dots,k_N} = \int_{\mathbb{I}^N} f(x)\, \omega_k(x)\, dx \tag{1}$$

are Walsh-Fourier coefficients of the function f.

We consider the rectangular partial sum of this series

$$S_n(x; f) = \sum_{k_1=0}^{n_1-1} \dots \sum_{k_N=0}^{n_N-1} c_k\, w_k(x), \quad n = (n_1, \dots, n_N) \in \mathbb{Z}_1^N,$$

whose particular case is the square partial sum $S_{n_0}(x; f)$, when $n_1 = \dots = n_N = n_0$.

Let E be an arbitrary measurable set, $E \subset \mathbb{I}^N$, $\mu E > 0$ ($\mu = \mu_N$ is the N-dimensional Lebesque measure), and let $\mathcal{F}(E)$ be a subspace of $L_1(E)$ such that the multiple Walsh-Fourier series (summed over rectangles) of any function $f \in \mathcal{F}(\mathbb{I}^N)$ converges a.e. on $\mathbb{I}^N$.

We study the behavior of $S_n(x; f)$ as $n \to \infty$, i.e. $\min_{1 \le j \le N} n_j \to \infty$ (or $S_{n_0}(x; f)$ as $n_0 \to \infty$) on $\mathbb{I}^N$ depending on the smoothness of the function f (i.e. on the type of the space $\mathcal{F}(\mathbb{I}^N)$) and on the structural and geometric characteristics of the set E.

3. Some Results on Convergence of Double Walsh-Fourier Series

For square summation the double Walsh-Fourier series (as follows from the result of F. Móricz, [8]) converges a.e. on $\mathbb{I}^2$ for functions in the class $L_2(\mathbb{I}^2)$, whereas for rectangular summation the double Walsh-Fourier series can diverge a.e. on $\mathbb{I}^2$ even for the continuous on $\mathbb{I}^2$ function (see the result of R.D.Getsadze [9]). From the theorem of E.M.Nikishin [10] concerning the Weyl multipliers (for convergence over rectangles of the double Fourier series with respect to the system of the form $\{\psi_{n_1}(x_1) \cdot \psi_{n_2}(x_2)\}_{n_1, n_2 = 1}^{\infty}$, where $\{\psi_{n_s}(x_s)\}_{n_s=1}^{\infty}$, $s = 1, 2$ is the orthonormal on a segment system of functions) and the result of P.Billard [11] concerning convergence of the one-dimensional Walsh-Fourier series of functions in L_2 it follows: if the following condition on Fourier coefficients (1) of the function $f \in L_2(\mathbb{I}^2)$ is true:

$$\sum_{k_1, k_2 = 0}^{\infty} |c_{k_1, k_2}|^2 \cdot \log^2[\min(|k_1|, |k_2|) + 2] < \infty, \tag{2}$$

then the double Walsh-Fourier series summed over rectangles of the function f converges a.e. on $\mathbb{I}^2$.

Let us note, that in solution of the problem (considered in the present paper) for one-dimensional Walsh-Fourier series we used (see section 1) the validity (for $N = 1$ in the class L_1) of the principle of the classical localization, which permits to state that for any open (nonempty) set $E \subset \mathbb{I}^1$ and for any function $f \in L_1(\mathbb{I}^1)$ such that $f(x) = 0$ on E

$$\lim_{n \to \infty} S_n(x; f) = 0 \quad \text{uniformly on any compact set} \quad K \subset E. \tag{3}$$

Unfortunately, for multiple (i.e. for $N \ge 2$) Fourier series (both with respect to the trigonometric system and to Walsh system) such localization is not true even for continuous functions (for more details see our papers [12], [13]).

Being in the framework of the classes $L_p(\mathbb{I}^N)$, $p \ge 1$ we “replaced” in (3) the uniform convergence by the convergence a.e., introducing the following concept of *the generalized localization almost everywhere* (see [14], [15][3]).

Let E, $E \subset \mathbb{I}^N$, $N \ge 1$ be an arbitrary set of positive measure. On the set E for multiple Fourier series of functions in the classes $L_p(\mathbb{I}^N)$, $p \ge 1$ *the generalized*

[3]In the paper [14] the concept of *the generalized localization a.e.* was introduced for trigonometric Fourier series.

localization almost everywhere is valid if for any function $f \in L_p(\mathbb{I}^N)$, $f(x) = 0$ on E the multiple Fourier series of the function f converges a.e. to zero on the set E.

In 1995 in [12] for $N = 2$ we proved the validity of the generalized localization a.e. for the double Walsh-Fourier series summed over rectangles on arbitrary open (open a.e.) set in the classes $L_p(\mathbb{I}^2)$, $p > 1$ (see [12, Theorem 1]).

Concerning the cases $N = 2$, $p = 1$ and $N > 2$, $p > 1$, in the same paper [12] (see also [15]) we ascertained the invalidity of the generalized localization a.e. in the indicated cases not only on the open sets, but also on any non-dense in $\mathbb{I}^N$ set.

Later in [16] (see also [17] and [18]) we (extending the notion of generalized localization a.e. on the Lebesgue-Orlicz classes) strengthened the result (of Theorem 1) of the paper [12], proving the following theorem

Theorem A. *Let E, $E \subset \mathbb{I}^2$ be an arbitrary open a.e. set, $\mu E > 0$. For any function $f \in L(\log^+ L)^2(\mathbb{I}^2)$, $f(x) = 0$ on E*

$$\lim_{n\to\infty} S_n(x; f) = 0 \quad \textit{almost everywhere on} \quad E.$$

Thus, for double Walsh-Fourier series summed over rectangles of the function in the classes $L(\log^+ L)^2(\mathbb{I}^2)$ the generalized localization a.e. is true on the open a.e. sets, but, as it was already said, the generalized localization a.e. is not true in the class $L_1(\mathbb{I}^2)$ on the wide class of sets, in particular, it is not true on the open sets.

Being again in the framework of classes $L_p(\mathbb{I}^N)$, $p \geq 1$, it was natural (the same way, as for the trigonometric system, see [15]) to pass to a more refined apparatus for studying the behavior of the Fourier series of a function f on the sets where f equals zero, namely, to the concept of "*the weak generalized localization a.e.*" (on the set E *the weak generalized localization almost everywhere* is true, if for any function $f \in L_p(\mathbb{I}^N)$, $f(x) = 0$ on E the multiple Fourier series of the function f converges a.e. to zero on some subset E_0, $E_0 \subset E$, $\mu E_0 > 0$).

In the paper [13] (see also [15], [18]) we obtained the criteria of the weak generalized localization a.e. in the class $L_1(\mathbb{I}^N)$, $N \geq 1$. For $N = 2$ let us formulate the particular case of this result (see [13, Theorem 2′]), and for this let us give the following definitions.

Let us consider on the axis Ox_j an arbitrary (nonempty) open set $\Omega_j \subset \mathbb{I}^1$, $j = 1, 2$, and denote as W^0 and W the sets

$$W^0 = \left(\Omega_1 \times \mathbb{I}^1\right) \cap \left(\mathbb{I}^1 \times \Omega_2\right) \tag{4}$$

and

$$W = W(W^0) = \left(\Omega_1 \times \mathbb{I}^1\right) \cup \left(\mathbb{I}^1 \times \Omega_2\right). \tag{5}$$

We shall say that a set E possesses property $\mathbb{B}_1$ if there exists a set W of the form (5) such that $\mu\,(W \setminus E) = 0$; property $\mathbb{B}_1$ is property $\mathbb{B}_1(W^0)$ if $W = W(W^0)$.

Further, let us denote by $pr_{(x_j)}\{P\}$ the orthogonal projection of the set P, $P \subset I^2$ onto the axis Ox_j, $j = 1, 2$; by $int\,(P)$ the set of interior points of P; by $\overline{P}$ the closure of the set P and by $Fr\,P$ the boundary of P.

Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$. Let us denote $G = \mathbb{I}^2 \setminus E$ and consider the following two conditions on $Fr\, E$:

$$\mu\,(G \setminus \overline{int\,(G)}\,) \quad = \quad 0,\,^4 \tag{6}$$

$$\mu_1\left(Fr\, pr_{(x_j)}\{int\,(G)\}\right) \quad = \quad 0, \quad j = 1,\, 2, \tag{7}$$

where $\mu = \mu_2$ is the measure on the plane, μ_1 is the measure on the line.

Theorem B. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, and let $G = \mathbb{I}^2 \setminus E$.*

1. *If for some set W^0 of the form* (4) *the set E possesses property $\mathbb{B}_1(W^0)$, then for any function $f \in L_1(\mathbb{I}^2)$, $f(x) = 0$ on E*

$$\lim_{n\to\infty} S_n(x; f) = 0 \quad \textit{almost everywhere on} \quad W^0.$$

2. *Let in addition the set E satisfy conditions* (6) *and* (7)*. If the set E does not possess property $\mathbb{B}_1$, then there exists a function $f^{(0)} \in L_1(\mathbb{I}^2)$, $f^{(0)}(x) = 0$ on E such that*

$$\overline{\lim_{n\to\infty}}\, |S_n(x; f^{(0)})| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2.$$

4. Main Results

In the present paper, basing on Theorem A, we have obtained the result which shows possibility "to localize on an open a.e. subset" $E \subset \mathbb{I}^2$ condition (2) of convergence a.e. on the whole cube $\mathbb{I}^2$ of double Walsh-Fourier series.

Let E, $E \subset \mathbb{I}^2$ be an arbitrary set of positive measure. Assume

$$\mathcal{F}(E) \quad = \quad \Bigg\{ f \in L_2(E) : \sum_{k_1,\, k_2 = 0}^{\infty} \left| \iint_E f(x_1,\, x_2)\, \omega_{k_1}(x_1)\, \omega_{k_2}(x_2)\, dx_1\, dx_2 \right|^2$$
$$\times \log^2[\min(|k_1|,\, |k_2|) + 2] < +\infty \Bigg\}.$$

Theorem 4.1. *Let E be an arbitrary open a.e. set, $E \subset \mathbb{I}^2$, $\mu E > 0$. For any function $f \in \mathcal{F}(E) \cap L_p(\mathbb{I}^2)$, $1 < p \leq 2$*

$$\lim_{n\to\infty} S_n(x; f) = f(x) \quad \textit{almost everywhere on} \quad E.$$

Further, taking into account geometry of the set $E \subset \mathbb{I}^2$, and basing on Theorem B, we can get the following result, which shows under what conditions it is possible "to localize on some subset" of the set E condition (2) (of convergence a.e. on the whole cube $\mathbb{I}^2$ of the double Walsh-Fourier series) in the case when on the whole $\mathbb{I}^2$ the function is in the class L_1 only.

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[4]In particular, the sets G such that $\mu\,\{int\, G\} = \mu\, G$ satisfy this condition; in it's turn the, last condition is true, for example, for an arbitrary open set.

Theorem 4.2. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, with conditions on the boundary $Fr\, E$ – (6) and (7), and let the set E have an open (nonempty) subset E^0. For any function $f \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$,*

$$\lim_{n\to\infty} S_n(x;f) = f(x) \quad \textit{almost everywhere on} \quad E^0 \subset W^0$$

if and only if the set E possesses property $\mathbb{B}_1(W^0)$, where

$$W^0 = \left(pr_{(x_1)}\{E^0\} \times \mathbb{I}^1\right) \cap \left(\mathbb{I}^1 \times pr_{(x_2)}\{E^0\}\right). \tag{8}$$

Remark 4.3. *In the part of sufficiency the result of Theorem 4.2 is true without the restrictions* (6) *and* (7).

Taking into account "more fine" structural and geometric characteristics of the sets E and E^0 (which appear in Theorem 4.2), it is possible to obtain the following result

Theorem 4.4. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, with conditions on the boundary $Fr\, E$ – (6) and (7), and let E^0 be an open (nonempty) subset of E. If the set E possesses property $\mathbb{B}_1(W^0)$, where the set W^0 is defined in* (8)*, but for any set $\widetilde{W}^0$ of the form* (4) *such that $\mu(\widetilde{W}^0 \setminus W^0) > 0$ the set E does not possess property $\mathbb{B}_1(\widetilde{W}^0)$, then*

1. *If $\mu(\mathbb{I}^2 \setminus W^0) > 0$, then there exists a function $f \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$ such that*

$$\overline{\lim_{n\to\infty}} |S_n(x;f)| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2 \setminus W^0.$$

2. *If $\mu(W^0 \setminus E^0) > 0$, and $\mu\, Fr\, E^0 = 0$, then there exists a function $f^{(1)} \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$ such that*

$$\overline{\lim_{n\to\infty}} |S_n(x;f^{(1)})| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2 \setminus E^0.$$

And finally, let us once more turn our attention to the questions of convergence a.e. of Walsh-Fourier series in the classes $\mathcal{F}(E) \cap L_p(\mathbb{I}^N)$, $p \geq 1$, $N \geq 1$ outside the set E (see section 1), this time for $N = 2$. Basing on the result concerning general properties of sequences of linear operators obtained by I.L.Bloshanskii in [19] (see [19, Theorem 1]) and using the function constructed by R.D.Getsadze in [9] we can get the following result.

Theorem 4.5. *For any closed set $E \subset \mathbb{I}^2$, $\mu E > 0$ there exists a function $f \in L_\infty(\mathbb{I}^2)$, $f(x) = 0$ on E such that*

1. $\lim_{n\to\infty} S_n(x;f) = 0$ *almost everywhere on* E,
2. $\overline{\lim_{n\to\infty}} |S_n(x;f)| = +\infty$ *almost everywhere on* $\mathbb{I}^2 \setminus E$.

Remark 4.6. *For any (nonempty) open set E, $E \subset \mathbb{I}^2$ with the boundary of measure zero the result similar to the result of Theorem 4.5 directly follows from* [9] *and* [15].

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Linear Transformations of $\mathbb{R}^N$ and Problems of Convergence of Fourier Series of Functions Which Equal Zero on Some Set

I.L. Bloshanskii

Abstract. Let $\mathfrak{M}$ be a class of (all) linear transformations of $\mathbb{R}^N$, $N \geq 1$. Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N$ be some linear subspace of $L_1(\mathbb{T}^N)$, and let $\mathfrak{A}$ be an arbitrary set of positive measure $\mathfrak{A} \subset \mathbb{T}^N$.

We consider the problem: how are the sets of convergence and divergence everywhere or almost everywhere (a.e.) of trigonometric Fourier series (in case $N \geq 2$ summed over rectangles) of function $(f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x))$, $f \in \mathcal{A}$, $f(x) = 0$ on $\mathfrak{A}$, $\mathfrak{m} \in \mathfrak{M}$, changed depending on the smoothness of the function f (i.e. on the space $\mathcal{A}$), as well as on the transformation $\mathfrak{m}$.

In the paper a (wide) class of spaces $\mathcal{A}$ is found such that for each $\mathcal{A}$ the system of classes (of nonsingular linear transformations) Ψ_k, $\Psi_k \subset \mathfrak{M}$ $(k = 0, 1, \ldots, N)$, which "change" the sets of convergence and divergence everywhere or a.e. of the indicated Fourier expansions is defined.

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1. Discussion of the Problem

In the theory of Fourier expansions the following problem plays an important role: how properties of Fourier expansions are affected by modifying the function that generates these expansions?

In the one-dimensional case to this range of problems, for example, the following result belongs obtained in 1940 by D.E.Men'shov [1] (for trigonometric Fourier series): any measurable function finite almost everywhere on $\mathbb{T}^1 = [-\pi, \pi)$ (in particular, any continuous function f, $f \in \mathbb{C}(\mathbb{T}^1)$), can be changed on a set of

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arbitrary small measure so that the obtained function has the uniformly convergent Fourier series.

Let us also note the classical problem posed by N.N.Luzin: does a continuous function exist such that, after the continuous transformation of variable, it becomes a function with absolutely convergent Fourier series. As it is known the answer to this question turned out to be negative: in 1981 A.M.Olevskii [2] proved the existence of a function f, $f \in \mathbb{C}(\mathbb{T}^1)$, such that for any homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, – the Fourier series of superposition $(f \circ \varphi)(x) = f(\varphi(x))$ is not absolutely convergent. Let us mention, as well, the result of 1935 by H.Bohr [3], who proved that for any continuous function f there exists a homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that the Fourier series of superposition $f \circ \varphi$ is uniformly convergent. (The detailed survey of the concerning results in the one-dimensional case see in the papers of J.P.Kahane [4] and A.M.Olevskii [5, 6].)

As for the multiple case, in 1998 A.A.Saakyan [7], generalizing the result of H.Bohr, proved that for any function $f \in \mathbb{C}(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N, N \geq 2$ (and therefore, for the continuous function with trigonometric Fourier series rectangularly divergent everywhere on $\mathbb{T}^N$ – see the example of Ch.Fefferman [8]), there exists a homeomorphism $\varphi : \mathbb{T}^N \to \mathbb{T}^N$ such that trigonometric Fourier series of superposition $f \circ \varphi$ uniformly rectangularly converges. The same year S.Galstyan and G.Karagulyan [9] proved an "opposite" (in a certain sense) result, namely: for any function $f \in \mathbb{C}(\mathbb{T}^N)$, $N \geq 2$ (which has no intervals of constancy in $\mathbb{T}^N$) there exists a homeomorphism $\varphi : \mathbb{T}^N \to \mathbb{T}^N$ such that the Fourier series of $f \circ \varphi$ rectangularly diverges almost everywhere (a.e.).

In 2000 O.S. Dragoshanskii [10] published the following result: there exists a function $f \in \mathbb{C}(\mathbb{T}^2)$ (whose support belongs to the square $[\frac{1}{2}, \frac{3}{4}]^2$) such that the double trigonometric Fourier series of f converges rectangularly a.e. on $\mathbb{T}^2$, while the same series but of the function $f \circ \tau$, where τ is a rotation of the coordinate system $\mathbb{R}^2$ on an angle $\frac{\pi}{4}$ diverges rectangularly on its support. In the same paper it was proved that rotation on the angle $\frac{\pi}{4}$ can "spoil", as well, the uniform convergence of the series under consideration.

We [11] in 2002 studied the problem concerning convergence everywhere and a.e. of multiple trigonometric Fourier series (summed over rectangles) of the function $f \circ \tau$, when $f \in L_1(\mathbb{T}^N)$, $N \geq 2$, $f(x) = 0$ on some subset (of positive measure) of $\mathbb{T}^N$, and τ is a rotation of the coordinate system $\mathbb{R}^N$ on an arbitrary angle.

In its turn, the results earlier obtained by us (see, e.g., [12]– [18]) which describe *the structural and geometric characteristics* (**SGC**) of sets of convergence and divergence a.e. and everywhere for multiple trigonometric Fourier series, multiple Walsh-Fourier series (summed over rectangles) and multiple Fourier integrals of functions f from various functional spaces $\mathcal{A}$ (e.g., L_1, Orlicz classes $L(\log^+ L)^s$, $s > 1$, the classes L_p, $1 < p < \infty$, $\mathbb{C}$, H^ω, etc.), f equals zero on some set $\mathfrak{A}$ of positive measure, permit to make some conclusions concerning convergence a.e. and everywhere of multiple Fourier expansions of the superposition $f \circ \psi$, when $f \in \mathcal{A}$, $f(x) = 0$ on $\mathfrak{A}$, and ψ belongs to some class Ψ of linear (e.g., orthogonal) transformations of $\mathbb{R}^N$, $\Psi \subset \mathfrak{M}$, where $\mathfrak{M}$ is the set of (all) linear transformations of $\mathbb{R}^N$.

It is a matter of interest to find (describe) all those classes of transformations Ψ, which for the given space $\mathcal{A}$ "change" the sets of convergence and divergence everywhere or a.e. of the multiple Fourier expansion of a function f in the space $\mathcal{A}$ ($f(x) = 0$ on some set of positive measure), i.e. to give description of pairs $(\mathcal{A}, \Psi)$.

2. Notation

Consider the N–dimensional Euclidean space $\mathbb{R}^N$, whose elements will be denoted as $x = (x_1, \ldots, x_N)$, and set $kx = k_1x_1 + \cdots + k_Nx_N$, $|x| = (x_1^2 + \ldots + x_N^2)^{1/2}$.

Let $\mathbb{Z}^N$, $\mathbb{Z}^N \subset \mathbb{R}^N$ be a set of all vectors with integer coordinates, let us also define the set $\mathbb{Z}^N_\alpha = \{n = (n_1, \ldots, n_N) \in \mathbb{Z}^N : n_j \geq \alpha,\ j = 1, \ldots, N\}$, $\alpha \in \mathbb{Z}^1$.

Let $S_n(x; f)$, $n \in \mathbb{Z}^N_1$, $N \geq 1$ be the rectangular partial sum of trigonometric Fourier series of a function $f \in L_1(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N$, whose particular case is the square partial sum $S_{n_0}(x; f)$, when $n_1 = \ldots = n_N = n_0$. Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^N)$ be some linear subspace of the space $L_1(\mathbb{T}^N)$, $\mathfrak{A}$ – an arbitrary measurable set, $\mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > 0$ ($\mu = \mu_N$ is the N-dimensional Lebesgue measure), and let $f(x) = 0$ on $\mathfrak{A}$.

We investigate how does the behavior of $S_n(x; f)$ as $n \to \infty$, i.e. $\min\limits_{1 \leq j \leq N} n_j \to \infty$ (or $S_{n_0}(x; f)$ as $n_0 \to \infty$) on $\mathbb{T}^N$ depend on the smoothness of the function f (i.e. on the type of the space $\mathcal{A}$), on the "modification" of the function f, and, finally, on *the structural and geometric characteristics* of the set $\mathfrak{A}$ (**SGC**($\mathfrak{A}$)).

3. Definition of the System of Functional Spaces

Denote as $\mathbb{F} = \mathbb{F}_N = \left\{\mathcal{A}_k^{(j)}\right\}_{k,j}$ a matrix $N \times 6$, whose elements are functional spaces $\mathcal{A}_k^{(j)} = \mathcal{A}_k^{(j)}(\mathbb{T}^N)$, $k = 1, \ldots, N$; $N \geq 1$ and $j \in \{0\} \bigcup J$, where $J = \{1, 2, \ldots, 5\}$. The spaces $\mathcal{A}_k^{(j)}$ will be defined as follows. For $k = 1, 2$ we set:

$$\mathcal{A}_1^{(j)} = \mathcal{A}_1^{(0)} = L_1\,,\ j \in J\,;\quad \mathcal{A}_2^{(0)} = \mathcal{A}_2^{(1)} = L_\infty\,;\quad \mathcal{A}_2^{(2)} = \mathcal{A}_2^{(3)} = L_2\,;$$
$$\mathcal{A}_2^{(4)} = L_p\,,\ 1 < p < 2;\quad \mathcal{A}_2^{(5)} = L(\log^+ L)^2. \tag{3.1}$$

For $k = 3, \ldots, N$ we set:

$$\mathcal{A}_k^{(0)} = H^{\overline{\omega}^{(k)}},$$

where $\overline{\omega}^{(k)}(\delta)$ is the modulus of continuity $\overline{\omega}^{(k)}(\delta) = \overline{\omega}_\lambda^{(k)}(\delta) = \lambda(\delta) \cdot \left(\log \frac{1}{\delta}\right)^{-\left[\frac{k}{2}\right]}$, where $[\xi]$ is the integral part of ξ, and $\lambda(\delta)$ is a function increasing to $+\infty$ as $\delta \to +0$ and $\lambda(\delta) = o(\log\log \frac{1}{\delta})$, $\delta \to +0$;

$$\mathcal{A}_k^{(1)} = H^{\omega^{(k-1)}} \quad \text{and} \quad \mathcal{A}_k^{(2)} = H_2^{\omega^{(k-1)}}\,,$$

where $\omega^{(k-1)}(\delta) = \omega_\varepsilon^{(k-1)}(\delta) = (\log \frac{1}{\delta})^{-\frac{k-1}{2}-\varepsilon}$, $0 < \varepsilon < \frac{1}{2}$;[1]

$$\mathcal{A}_k^{(3)} = \left\{ f \in L_2(\mathbb{T}^N) : \sum_{n\in\mathbb{Z}^N} |c_n|^2 \cdot \max_{1\le j_1<\ldots<j_{k-1}\le N} \prod_{s=1}^{k-1} \log(|n_{j_s}|+2) < +\infty \right\},$$

where $c_n = c_n(f)$ are Fourier coefficients of function f; and, besides, we set

$$\mathcal{A}_3^{(4)} = \left\{ f \in L_2(\mathbb{T}^N) : \sum_{n\in\mathbb{Z}^N} |c_n|^2 \log^2[\max_{s,l=1,2,\ldots,N} \min_{s\neq l}(|n_s|,|n_l|)+2] < +\infty \right\},$$

$$\mathcal{A}_3^{(5)} = H^{\omega^{(1)}}, \quad \text{where} \quad \omega^{(1)}(\delta) = o([\log\frac{1}{\delta}\log\log\log\frac{1}{\delta}]^{-1}), \quad \delta \to +0.$$

For $k = 4, \ldots, N$ we set:

$$\mathcal{A}_k^{(4)} = \mathcal{A}_k^{(5)} = \mathcal{A}_k^{(1)}.$$

Let us note that "smoothness" of functions $f \in \mathcal{A}_k^{(j)}(\mathbb{T}^N)$ certainly "increases" with the growth of the number k, i.e. $\mathcal{A}_k^{(j)} \supset \mathcal{A}_{k+1}^{(j)}$, $j \in \{0\} \bigcup J$, $k = 1\ldots, N-1$.

Let us also note that the classes $\mathcal{A}_k^{(j)}(\mathbb{T}^N)$, $j \in J$ have the following property: in the case $k > 1$ for any function $f \in \mathcal{A}_k^{(j)}(\mathbb{T}^{k-1})$ convergence of $(k-1)$ -multiple trigonometric Fourier series summed over rectangles takes place a.e. on $\mathbb{T}^{k-1}$ (see results of L.Carleson [19], R.Hunt [20] ($k = 2$); K.I.Oskolkov [21], P.Sjölin [22] ($k = 3$); L.V.Zhizhiashvili [23] and [24], F.Móricz [25] ($k \ge 4$)).

The indicated ("functional") matrix $\mathbb{F}$ was introduced by us in the paper [26].

4. Definition of the Classes of Linear Transformations of $\mathbb{R}^N$

Let $\mathfrak{M}$ be a class of (all) linear transformations of $\mathbb{R}^N$, $N \ge 1$. Denote as Ψ_1, $\Psi_1 \subset \mathfrak{M}$ the class of linear nonsingular transformations, whose inverse transformations have matrices $\mathbb{A} = \{a_{l,m}\}_{l,m=1}^N$, satisfying condition: there exists s, $1 \le s \le N$ such that

$$\max_{1\le l\le N} |a_{l,s}| < 1. \tag{4.1}$$

Further, in the case of dimension of the space $N \ge 2$, we define the following N subsets of Ψ_1.

First, for any k, $2 \le k \le N$, we define the class of transformations Ψ_k: $\psi \in \Psi_k$ if the matrix $\mathbb{A}$ of inverse (to ψ) transformation ψ^{-1} satisfies condition: there exist $m_1, \ldots, m_k$, $1 \le m_1 < \ldots < m_k \le N$ such that

$$\max_{1\le l\le N} \{|a_{l,m_1}| + \ldots + |a_{l,m_k}|\} < 1. \tag{4.2}$$

For classes of transformations $\Psi_1, \ldots, \Psi_N$, the embeddings are obvious: $\Psi_1 \supset \Psi_2 \supset \ldots \supset \Psi_N$.

[1]Note that $\mathcal{A}_k^{(0)} \subset \mathcal{A}_k^{(1)}$ if k is even, $k \ge 4$.

Second, for $N \geq 2$ we define the class of transformations $\Psi_0 \subset \Psi_1$. Let $\mathcal{F}$ be a group of rotations of $\mathbb{R}^N$ about the origin, and let $\mathcal{F}_0$ be a set of rotations from $\mathcal{F}$, that are compositions of rotations in all the two-dimensional coordinate planes by angles which are integer multiple of $\frac{\pi}{2}$. Set[2]

$$\Psi_0 = \mathcal{F} \setminus \mathcal{F}_0. \tag{4.3}$$

5. Setting of the Problem and Approaches to Its Solution

We pose and study the problem: how are the sets of convergence and divergence (everywhere or a.e.) of trigonometric Fourier series (in case $N \geq 2$ summed over rectangles) of function f, belonging to one of the spaces $\mathcal{A}$ (elements of the matrix $\mathbb{F}$) and vanishing on some measurable set $\mathfrak{A} \subset \mathbb{T}^N$, $0 < \mu\mathfrak{A} < (2\pi)^N$, $N \geq 1$, ($\mu = \mu_N$ is the Lebesgue measure) changed (if changed) in dependence on the transformation $\psi \in \Psi$, where $\Psi = \Psi_k$, $0 \leq k \leq N$? Thus, we want to "describe" a pair $(\mathcal{A}, \Psi)$.

Further, for any set $E \subset \mathbb{R}^N$ and any $\mathfrak{m} \in \mathfrak{M}$ we define the set $\mathfrak{m}(E) = \{y \in \mathbb{R}^N : y = \mathfrak{m}(x),\ x \in E\}$. Analogously the set $\mathfrak{m}^{-1}(E)$ is defined, where transformation $\mathfrak{m}^{-1}$ is such that: $\mathfrak{m}^{-1} \cdot \mathfrak{m} = 1$ (if $\mathfrak{m}^{-1}$ exists). It is obvious that for any $E \subset \mathbb{T}^N$ there exists $\mathfrak{m} \in \mathfrak{M}$ such that $\mathfrak{m}(E) \not\subset \mathbb{T}^N$.

Thus, taking into account that (in the present paper) we consider 2π-periodic functions $f(x)$, the question arises: how the Fourier series should be understood for the function $(f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x))$, e.g., for rotation (of the coordinate system of $\mathbb{R}^N$), i.e. when $\mathfrak{m} = \tau \in \mathcal{F}$.[3]

Analogously to the paper [11], where we considered the group of rotations $\mathcal{F}$, we shall formulate two variants how the Fourier series of function $f \circ \mathfrak{m}$, $\mathfrak{m} \in \mathfrak{M}$ can be understood.

Let us fix an arbitrary $\mathfrak{m} \in \mathfrak{M}$. For any function $f \in L_1(\mathbb{T}^N)$[4] we define 2π-periodic (for $N \geq 2$ — in each argument) functions $g^{(l)}_{\mathfrak{m}}(x)$, $l = 1, 2$, so that on $\mathbb{T}^N$ these functions are defined by equalities:

$$g^{(1)}_{\mathfrak{m}}(x) = (f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x)), \quad x \in \mathbb{T}^N, \tag{5.1}$$

$$g^{(2)}_{\mathfrak{m}}(x) = (f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x)) \cdot \chi_{_{\mathbb{T}^N}}(\mathfrak{m}(x)), \quad x \in \mathbb{T}^N, \tag{5.2}$$

where $\chi_{_{\mathbb{T}^N}}(\cdot)$ is the characteristic function of the cube $\mathbb{T}^N$.

Thus, the posed above problem is decomposed into two problems in dependence on the regard to Fourier series of function $f \circ \mathfrak{m}$. Further in the text: for $l = 1$ – the problem 1, and for $l = 2$ – the problem 2.

Earlier we have investigated [12]–[18] (see also [26]) the problem concerning changes of the structure and geometry of sets of convergence and divergence a.e.

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[2]It is obvious that rotations $\tau \in \mathcal{F}_0$ can not change the sets of convergence or divergence of multiple Fourier expansions.

[3]Let us note that for Fourier integrals $\int_{\mathbb{R}^N} \widehat{h}(\xi) e^{ix\xi}\, d\xi$, $x \in \mathbb{R}^N$, of function $h \in L_1(\mathbb{R}^N)$ the problem "in this sense" does not arise.

[4]Naturally, the function $f(x)$ is 2π -periodic in each argument.

and everywhere for (multiple) trigonometric Fourier series (for $N \geq 2$ summed over rectangles) of functions f in $\mathcal{A}_k^{(j)}(\mathbb{T}^N)$, $k = 1, \ldots, N$, $j \in \{0\} \bigcup J$, $f(x) = 0$ on some measurable set $\mathfrak{A} \subset \mathbb{T}^N$, in dependence on changes of structure and geometry of the set $\mathfrak{A}$. So, the both posed problems are reduced (in fact) to the study of the question concerning changes of structure and geometry of sets $\psi^{-1}(\mathfrak{A}) \cap \mathbb{T}^N$ and $\mathbb{T}^N \setminus supp(f \circ \psi)$, in dependence on $\psi \in \Psi_k$, $0 \leq k \leq N$.[5]

Let us note that problem 1, being a more complicated problem, is, at the same time, a more natural one for trigonometric Fourier series even in the study of such "unnatural" for these series "problem of rotations".

Let us show some particular solutions of problem 2, whose results give the description of the pairs $(\mathcal{A}, \Psi)$, more exactly, let us formulate the results describing (some) relation between the "smoothness" (in terms of the matrix $\mathbb{F}$) of the function f ($f(x) = 0$ on $\mathfrak{A}$) and the transformation ψ (in terms of the classes Ψ_k).

6. The Set of Transformations Ψ_k, $k = 1, \ldots, N$. Solution of Problem 2

Two following theorems give description of the pair $(\mathcal{A}_1^{(j)}, \Psi_1)$, $j \in \{0\} \bigcup J$, i.e., taking account of (3.1), – the pair (L_1, Ψ_1) (for $N = 1$ and for $N > 1$, respectively).

Theorem 6.1. *For any $\psi \in \Psi_1$ and ε, $0 < \varepsilon < 2\pi$, there exist the measurable sets $\Omega = \Omega(\varepsilon, \psi) \subset \mathbb{T}^1$, $\mathfrak{A} = \mathfrak{A}(\varepsilon, \psi) \subset \mathbb{T}^1$: $\mu\Omega > 0$, $\mu\mathfrak{A} > 2\pi - \varepsilon$ and a function $f = f_{\varepsilon, \psi} \in L_1(\mathbb{T}^1)$, $f(x) = 0$ on $\mathfrak{A}$, such that*

1. $$\varlimsup_{n \to \infty} |S_n(x; f)| = +\infty \quad \textit{in each point} \quad x \in \mathbb{T}^1, \tag{6.1}$$
2. $$\lim_{n \to \infty} S_n(x; f \circ \psi) = 0 \quad \textit{in each point} \quad x \in \Omega. \tag{6.2}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), *i.e.* $f \circ \psi = g_\psi^{(2)}$.

Theorem 6.2. *Let $N > 1$. For any $\psi \in \Psi_1$ and ε, $0 < \varepsilon < (2\pi)^N$, there exist the open sets $\Omega = \Omega(\varepsilon, \psi)$, $\mathfrak{A} = \mathfrak{A}(\varepsilon, \psi) : \Omega \subset \mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > (2\pi)^N - \varepsilon$, $0 < \mu\Omega < \mu\mathfrak{A}$ such that*

1. *There exists a function $f^{(0)} = f^{(0)}_{\varepsilon, \psi} \in L_1(\mathbb{T}^N)$, $f^{(0)}(x) = 0$ on $\mathfrak{A}$, and*
$$\varlimsup_{n_0 \to \infty} |S_{n_0}(x; f^{(0)})| = +\infty \quad \textit{in each point} \quad x \in \mathbb{T}^N. \tag{6.3}$$
2. *For any function $f \in L_1(\mathbb{T}^N)$, $f(x) = 0$ on $\mathfrak{A}$,*
$$\lim_{n \to \infty} S_n(x; f \circ \psi) = 0 \quad \textit{in each point} \quad x \in \Omega. \tag{6.4}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), *i.e.* $f \circ \psi = g_\psi^{(2)}$.

Analogous results are obtained for other pairs $(\mathcal{A}_r^{(j)}, \Psi_k)$, where $k \leq r \leq N$, $j \in \{0\} \bigcup J$ for $k = r = N$, if $N = 2$, and for $1 \leq k \leq 2 \cdot \left[\frac{N-1}{2}\right]$, if $N \geq 3$, namely, the following theorems are true

[5]Note that for singular transformations $\mathfrak{m} \in \mathfrak{M}$ the discussed problem becomes trivial.

Theorem 6.3. *For any $\psi \in \Psi_2$ and ε, $0 < \varepsilon < (2\pi)^2$, there exist the measurable sets $\Omega = \Omega(\varepsilon, \psi) \subset \mathbb{T}^2$, $\mathfrak{A} = \mathfrak{A}(\varepsilon, \psi) \subset \mathbb{T}^2 : \mu\Omega > 0$, $\mu\mathfrak{A} > (2\pi)^2 - \varepsilon$ and a function $f = f_{\varepsilon,\psi} \in L_\infty(\mathbb{T}^2)$, $f(x) = 0$ on $\mathfrak{A}$, such that*

1. $$\overline{\lim_{n\to\infty}} |S_n(x; f)| = +\infty \quad \text{for almost all} \quad x \in \mathbb{T}^2, \tag{6.5}$$
2. $$\lim_{n\to\infty} S_n(x; f \circ \psi) = 0 \quad \text{for almost all} \quad x \in \Omega. \tag{6.6}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), i.e. $f \circ \psi = g_\psi^{(2)}$.

Theorem 6.4. *Let $N \geq 3$. For any integer k and $r : 1 \leq k \leq 2 \cdot \left[\frac{N-1}{2}\right]$, $k \leq r \leq N$, and any ψ, $\varepsilon :$ $\psi \in \Psi_k$ and $0 < \varepsilon < (2\pi)^N$, there exist the open sets $\mathfrak{A} = \mathfrak{A}_k(\varepsilon, \psi)$, $\Omega = \Omega_{k,r}(\varepsilon, \psi) : \Omega \subset \mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > (2\pi)^N - \varepsilon$, $0 < \mu\Omega < \mu\mathfrak{A}$ such that*

1. *There exists a function $f^{(0)} = f^{(0)}_{k,\varepsilon,\psi} \in \mathcal{A}_k^{(0)}(\mathbb{T}^N)$, $f^{(0)}(x) = 0$ on $\mathfrak{A}$, and*
$$\overline{\lim_{n\to\infty}} |S_n(x; f^{(0)})| = +\infty \quad \text{for almost all} \quad x \in \mathbb{T}^N. \tag{6.7}$$
2. *For any $j \in J$ and for any function $f \in \mathcal{A}_r^{(j)}(\mathbb{T}^N)$, $f(x) = 0$ on $\mathfrak{A}$,*
$$\lim_{n\to\infty} S_n(x; f \circ \psi) = 0 \quad \text{for almost all} \quad x \in \Omega. \tag{6.8}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), i.e. $f \circ \psi = g_\psi^{(2)}$ and $[\xi]$ is the integral part of ξ.

Remark 6.5. Taking into account the embeddings: $\mathcal{A}_k^{(j)} \supset \mathcal{A}_r^{(j)}$, $j \in \{0\} \bigcup J$ for all k and $r : k \leq r \leq N$, and embeddings $\mathcal{A}_k^{(0)} \subset \mathcal{A}_k^{(1)}$ for even k, $k \geq 4$, we can make a conclusion that the 2-nd point in theorem 6.4 (see estimate (6.8)) is true, as well, (in case of even k, $k \geq 4$) for the function $f^{(0)} \in \mathcal{A}_k^{(0)}$, defined in point 1 of this theorem (see estimate (6.7)).

Further, in theorems 6.1– 6.4 the function f (in estimates (6.1), (6.2), (6.5) and (6.6)) and the function $f^{(0)}$ (in estimates (6.3) and (6.7)) depended on the transformation ψ, $\psi \in \Psi_k$ (i.e. for each transformation ψ its own function f was found (the function $f^{(0)}$)). So the question naturally arises: is it possible to construct a “universal” function f ($f \in \mathcal{A}_k^{(j)}$, with arbitrary small support), whose Fourier series unrestrictedly diverges (everywhere or a.e.) on $\mathbb{T}^N$, and, at the same time, the same series, but already of the function $f \circ \psi$, for any $\psi \in \Psi_k$ converges (everywhere or a.e.) on some subset (of positive measure) of $\mathbb{T}^N$?

For the class of transformations $\Psi_0 \subset \mathcal{F}$, where $\mathcal{F}$ is a group of rotations of $\mathbb{R}^N$ about the origin (see condition (4.3)), the answer to the posed question appears to be positive in many cases, and this fact is connected (partially) with the definite “specific character” of the problem (considered in the present paper) for transformations in $\mathcal{F}$.

7. The Group of Rotations of $\mathbb{R}^N$ About the Origin

Denote as

$$\mathbb{B}(O,\pi) = \{x \in \mathbb{R}^N : |x| < \pi\} \tag{7.1}$$

a ball inscribed into the cube $\mathbb{T}^N$. It is obvious that for any rotation τ, $\tau \in \mathcal{F}$ and for any set $E \subset \mathbb{B}(O,\pi)$ – the set $\tau(E) \subset \mathbb{T}^N$.

Note that for $\tau \in \mathcal{F}$ the set $\tau^{-1}(\mathbb{T}^N) \setminus \mathbb{T}^N$ (and, moreover, the set $\{x \in \mathbb{R}^N : |x| \le \sqrt{N}\cdot\pi\} \setminus \mathbb{T}^N$) in the definition of functions $g_\tau^{(l)}$, $l = 1,2$ (see (5.1), (5.2)) – (in general) "vanishes". The last is natural for the problem under consideration for Fourier series. However, this "vanishing" can be avoided in the problem 2 if we consider a more "narrow" class of functions $f(x)$, namely, f: *supp* $(f \cdot \chi_{\mathbb{T}^N}) \subset \mathbb{B}(O,\pi)$.[6]

7.1. The Set of Transformations Ψ_0. Solution of Problem 2. Functions in L_1

Theorem 7.1. *Let $\mathcal{F}$ be a group of rotations of $\mathbb{R}^2$ and let Ψ_0, $\Psi_0 \subset \mathcal{F}$ satisfy condition* (4.3). *For any ε, $0 < \varepsilon < (2\pi)^2$, there exists an open set $\mathfrak{A} = \mathfrak{A}_\varepsilon \subset \mathbb{T}^2$, $\mu\mathfrak{A} > (2\pi)^2 - \varepsilon$ such that*

1. *There exists a function $f^{(0)} = f_\varepsilon^{(0)} \in L_1(\mathbb{T}^2)$ such that $f^{(0)}(x) = 0$ on $\mathfrak{A}$, supp $(f^{(0)} \cdot \chi_{\mathbb{T}^2}) \subset \mathbb{B}(O,\pi)$ (see* (7.1) *) and*

$$\overline{\lim_{n_0\to\infty}} |S_{n_0}(x; f^{(0)})| = +\infty \quad \text{in each point} \quad x \in \mathbb{T}^2.$$

2. *For any $\tau \in \Psi_0$ there exists an open (nonempty) set $\Omega = \Omega_\varepsilon(\tau) \subset \mathfrak{A}$ such that for any function $f \in L_1(\mathbb{T}^2) : f(x) = 0$ on $\mathfrak{A}$, supp $(f \cdot \chi_{\mathbb{T}^2}) \subset \mathbb{B}(O,\pi)$,*

$$\lim_{n\to\infty} S_n(x; f\circ\tau) = 0 \quad \text{in each point} \quad x \in \Omega.$$

Here the notation $f\circ\tau$ is understood in the sense of equality (5.2), *i.e. $f\circ\tau = g_\tau^{(2)}$.*

Concerning the case $N \ge 3$, a "weaker", to some extent, result is true, for example, the following theorem holds.

Theorem 7.2. *Let $\mathcal{F}$ be a group of rotations of $\mathbb{R}^N$, $N \ge 3$, and let Ψ_0, $\Psi_0 \subset \mathcal{F}$ satisfy condition* (4.3). *For any j, $1 \le j \le N$, and any ε, $0 < \varepsilon < (2\pi)^N$, there exists an open set $\mathfrak{A} = \mathfrak{A}_{\varepsilon,j} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > (2\pi)^N - \varepsilon$ such that*

1. *There exists a function $f^{(0)} = f_\varepsilon^{(0)} \in L_1(\mathbb{T}^N)$ such that $f^{(0)}(x) = 0$ on $\mathfrak{A}$, supp $(f^{(0)} \cdot \chi_{\mathbb{T}^N}) \subset \mathbb{B}(O,\pi)$ (see* (7.1) *) and*

$$\overline{\lim_{n_0\to\infty}} |S_{n_0}(x; f^{(0)})| = +\infty \quad \text{in each point} \quad x \in \mathbb{T}^N. \tag{7.2}$$

---

[6] In the problem 1 (for $\tau \in \mathcal{F}$) this "vanishing" can be also avoided by choosing a "smaller" *supp* $(f \cdot \chi_{\mathbb{T}^N})$ but (as it is not difficult to calculate) only for $N = 2$ and $N = 3$.

2. *For any* $\tau \in \Psi_0$ *(except rotations about the axes containing the coordinate axis* Ox_j *) there exists an open (nonempty) set* $\Omega = \Omega_{\varepsilon,\,j}(\tau) \subset \mathfrak{A}$ *such that for any function* $f \in L_1(\mathbb{T}^N) : f(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f \cdot \chi_{_{\mathbb{T}^N}}) \subset \mathbb{B}(O, \pi)$,
$$\lim_{n\to\infty} S_n(x; f\circ\tau) = 0 \quad \textit{in each point} \quad x \in \Omega.$$

Here the notation $f\circ\tau$ *is understood in the sense of equality* (5.2), *i.e.* $f\circ\tau = g_\tau^{(2)}$.

We formulated Theorem 7.2 where we excluded rotations about the axes which are s-dimensional manifolds ($1 \le s \le N-2$) containing some (fixed) coordinate axis. Note that divergence of the "original" Fourier series (of function $f,\ f = f^{(0)}$ such that the support of the function *supp* $(f \cdot \chi_{_{\mathbb{T}^N}})$ in $\mathbb{B}(O,\pi)$) takes place on the whole cube $\mathbb{T}^N$(see (7.2)). If in the research of the posed problem (problem 2) we have no aim to have divergence of the (original) Fourier series on the whole cube $\mathbb{T}^N$, but we want to "remain" in the class of functions $\{f \in L_1 :\ supp\,(f \cdot \chi_{_{\mathbb{T}^N}}) \subset \mathbb{B}(O,\pi)\}$, we can obtain the following results.

Theorem 7.3. *Let* $\mathcal{F}$ *be a group of rotations of* $\mathbb{R}^N$, $N \ge 3$, *and let* $\Psi_0,\ \Psi_0 \subset \mathcal{F}$ *satisfy condition* (4.3). *For any* ε *and* $\alpha : 0 < \varepsilon < (2\pi)^N,\ 0 < \alpha < (2\pi)^N$, *there exist the open sets* $\Omega_0 = \Omega_0(\varepsilon, \alpha)$, $\mathfrak{A} = \mathfrak{A}_{\varepsilon,\,\alpha} : \Omega_0 \subset \mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > (2\pi)^N - \varepsilon$, $0 < \mu\Omega_0 < \min(\alpha, \mu\mathfrak{A})$ *such that*

1. *There exists a function* $f^{(0)} = f^{(0)}_{\varepsilon,\,\alpha} \in L_1(\mathbb{T}^N)$ *such that* $f^{(0)}(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f^{(0)} \cdot \chi_{_{\mathbb{T}^N}}) \subset \mathbb{B}(O,\pi)$ *(see* (7.1) *) and*
   a) $\overline{\lim\limits_{n_0\to\infty}} |S_{n_0}(x; f^{(0)})| = +\infty$ *in each point* $x \in \mathbb{T}^N \setminus \Omega_0$,
   b) $\lim\limits_{n\to\infty} S_n(x; f^{(0)}) = 0$ *in each point* $x \in \Omega_0$.
2. *For any* $\tau \in \Psi_0$ *there exists an open set* $\Omega_1 = \Omega_1(\tau, \varepsilon, \alpha) \subset \mathfrak{A}$ *such that* $\Omega_0 \subset \Omega_1$, $\mu\Omega_0 < \mu\Omega_1$ *and for any function* $f \in L_1(\mathbb{T}^N) : f(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f \cdot \chi_{_{\mathbb{T}^N}}) \subset \mathbb{B}(O,\pi)$,
$$\lim_{n\to\infty} S_n(x; f\circ\tau) = 0 \quad \textit{in each point} \quad x \in \Omega_1.$$

Here the notation $f\circ\tau$ *is understood in the sense of equality* (5.2), *i.e.* $f\circ\tau = g_\tau^{(2)}$.

7.2. The Set of Transformations Ψ_0. Solution of Problem 2. Functions in $L_p,\ p > 1$

Let us fix an arbitrary $\delta,\ 0 < \delta < 1$ and define the following set of "small oscillations (small rotations)" $\Psi_0(\delta),\ \Psi_0(\delta) \subset \Psi_0 :$
$$\Psi_0(\delta) = \left\{\tau \in \Psi_0 :\ |\tau x - x| < \delta \quad \text{for any} \quad x \in \mathbb{T}^N\right\}. \tag{7.3}$$

Theorem 7.4. *Let* $\mathcal{F}$ *be a group of rotations of* $\mathbb{R}^3$, *and let* $\Psi_0,\ \Psi_0 \subset \mathcal{F}$ *satisfy condition* (4.3). *There exist the open sets* $\Omega_0,\ \mathfrak{A} : \Omega_0 \subset \mathfrak{A} \subset \mathbb{T}^3$, $\mu\mathfrak{A} > (2\pi)^3 - \varepsilon$, $0 < \mu\Omega_0 < \mu\mathfrak{A}$ *such that*

1. *There exists a function* $f^{(0)} \in L_\infty(\mathbb{T}^3)$, *such that* $f^{(0)}(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f^{(0)} \cdot \chi_{_{\mathbb{T}^3}}) \subset \mathbb{B}(O,\pi)$ *(see* (7.1) *) and*

a) $\overline{\lim\limits_{n\to\infty}} |S_n(x; f^{(0)})| = +\infty$ *for almost all* $x \in \mathbb{T}^3 \setminus \Omega_0$,

b) $\lim\limits_{n\to\infty} S_n(x; f^{(0)}) = 0$ *for almost all* $x \in \Omega_0$.

2. *There exists* $\delta > 0$ *such that for any* $\tau \in \Psi_0(\delta) \subset \Psi_0$ (*see* (7.3)) *there exists an open set* $\Omega_1 = \Omega_1(\tau) \subset \mathfrak{A}$ *such that :* $\Omega_0 \cap \Omega_1 \neq \emptyset$, $\mu\Omega_0 < \mu\Omega_1$ *and for any function* $f \in L_\infty(\mathbb{T}^3) : f(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f \cdot \chi_{_{\mathbb{T}^3}}) \subset \mathbb{B}(O, \pi)$,

$$\lim_{n\to\infty} S_n(x; f \circ \tau) = 0 \quad \textit{for almost all} \quad x \in \Omega_1.$$

Here the notation $f \circ \tau$ *is understood in the sense of equality* (5.2), *i.e.* $f \circ \tau = g_\tau^{(2)}$.

7.3. The Set of Transformations Ψ_0. Solution of Problem 1

An analog of Theorem 7.1 is true, exactly

Theorem 7.5. *Let* $\mathcal{F}$ *be a group of rotations of* $\mathbb{R}^2$, *and let* Ψ_0, $\Psi_0 \subset \mathcal{F}$ *satisfy condition* (4.3). *For any* ε, $0 < \varepsilon < (2\pi)^2$ *there exist an open set* $\mathfrak{A} = \mathfrak{A}_\varepsilon \subset \mathbb{T}^2$, $\mu\mathfrak{A} > (2\pi)^2 - \varepsilon$ *such that*

1. *There exists a function* $f^{(0)} = f_\varepsilon^{(0)} \in L_1(\mathbb{T}^2)$ *such that* $f^{(0)}(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f^{(0)} \cdot \chi_{_{\mathbb{T}^2}}) \not\subset \mathbb{B}(O, \pi)$ (*see* (7.1)) *and*

$$\overline{\lim_{n_0\to\infty}} |S_{n_0}(x; f^{(0)})| = +\infty \quad \textit{in each point} \quad x \in \mathbb{T}^2.$$

2. *For any* $\tau \in \Psi_0$ *there exists an open (nonempty) set* $\Omega = \Omega_\varepsilon(\tau) \subset \mathfrak{A}$ *such that for any function* $f \in L_1(\mathbb{T}^2) : f(x) = 0$ *on* $\mathfrak{A}$, *supp* $(f \cdot \chi_{_{\mathbb{T}^2}}) \not\subset \mathbb{B}(O, \pi)$,

$$\lim_{n\to\infty} S_n(x; f \circ \tau) = 0 \quad \textit{in each point} \quad x \in \Omega.$$

Here the notation $f \circ \tau$ *is understood in the sense of equalities* (5.1), *i.e.* $f \circ \tau = g_\tau^{(1)}$.

Remark 7.6. The results of theorems 7.1, 7.2 and 7.5 in somewhat "more weak variant", where the 2-nd point was "formulated for the function $f^{(0)}$", were proved by us in [11].

Remark 7.7. The result of theorem 7.4 is valid as well for continuous functions and, moreover, for continuous functions with some modulus of continuity.

Remark 7.8. In theorems 6.2, 6.4, 7.1–7.3, and 7.5 the sets of convergence (to zero) are a finite collection of open rectangles (for $N \geq 3$ - parallelepipeds) with sides ($N \geq 3$ - edges) parallel to the coordinate axes.

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Sidon Type Inequalities for Wavelets

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Abstract. In 1938, S.Sidon [9] proved an inequality for the complex trigonometric system on interval [0, 1) known as Sidon type inequality. This inequality was generalized by Bojanic and Stanojevic [1]. The Walsh case was investigated by Moricz and Schipp [7]. Another generalization for trigonometric case was given by Buntinas and Tanovic-Miller [2]. Here in this paper we proved it for wavelet case and also obtained the convergence of wavelet series in L^1 norm.

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1. Introduction

Denote $u = \{u_n, n \in N \, or \, n \in Z\}$ an orthonormal system defined on interval I, where N is the set of non-negative integers and Z is the set of integers. The Dirchlet kernels with respect to this system are denoted by

$$D_n = D_n^u = \sum_{|k|<n} u_k \qquad (n \in p = N \setminus \{0\})$$

If $U = T$ is the complex trigonometric system on [0,1) i.e., $U_n(x) = e_n(x) = e^{2\Pi ix}$ $(x \in I, n \in Z)$ then for every sequence $a = \{a_n, n \in P\}$

$$M_n(a, U) = \frac{1}{n} \int_I \left| \sum_{k=0}^{n-1} a_k D_k(x) \right| dx \leq C \max_{0 \leq k < n} |a_k|, \tag{1.1}$$

holds with an absolute constant $C > 0$. This was proved by S.Sidon in 1938 [9]. Concerning the history and elegant proof see Telyakovskii [10].

A generalization of (1.1) is the following inequality

$$M_n(a,U) \le C_p \left(\frac{1}{n}\sum_{k=0}^{n-1}|a_k|^p\right)^{1/p} \qquad (p>1,\ n\in P)\,, \tag{1.2}$$

where C_p depends only on p.

For the trigonometric system it was proved by Bojanic and Stanojevic [1]. The same holds for complex trigonometric system $T = (e_n,\, n\in N)$ with non- negative indices which can be proved in the same way . The Walsh case was investigated by Moricz and Schipp [7] and (1.2) is proved for the Walsh - Paley system with constant $C_p = \frac{2.05p}{(p-1)}$.

Another generalization for the trigonometric case is given by Buntinas and Tanovic - Miller [2] namely, if $1\le p<2$ and $\frac{1}{p}+\frac{1}{q}=1$, then

$$\frac{1}{n}\int_I \left|\sum_{k=0}^{n-1} a_k D_k(x)\right| dx \le K_p \left\langle \left(\frac{\log\alpha}{n}\right)\sum_{k=1}^{n}|a_k| + \alpha^{-\frac{1}{q}}\left(\frac{1}{n}\sum_{k=1}^{n}|a_k|^p\right)^{1/p}\right\rangle$$

for all $\alpha\ge 1$ where $K_p = 2\left(1+e^{1/q}\right)(p-1)^{-1/p}$.

Let X be normed subspace of $L^1 = L^1[0,1)$ containing the dyadic step function, i.e., the function of the form

$$A_n = \sum_{k=0}^{2^n-1} a_k\chi[k2^{-n},(k+1)\,2^{-n}) \tag{1.3}$$

where χ_j denotes the characteristic function of the set 'j' and $a_0 = 0$. The norm in X is denoted by $\|a\|_X$. If $X = L^p$ $(1\le p\le\infty)$ then we denote L^p - norm by $\|.\|_p$. By means of this norm we introduce a sequence norm setting

$$\|a\|_X = \sup_{n\in N}\|A_n\|_X\,, \tag{1.4}$$

where the functions A_n are defined by (1.3). We denote by X^0 the set of sequences $a = \{a_k\}$ satisfying $\|a\|_X < \infty$. Using this notion (1.2) can be written in the form

$$M_n(a,U) = \sup_{n\in N} M_{2^n}(a,U) \le C_p\,\|a\|_p \quad (1\le p\le\infty)\,,$$

where $\|.\|_p$ denotes the sequence norm induced by the usual L^p -norm.

The dyadic Hardy space H^p can be defined by means of dyadic maximal function

$$f^*(x) = \sup_{n\in N}\frac{1}{|I_n(x)|}\int_{I_n(x)} |f(t)\,dt| \quad (x\in[0,1)\,,\, f\in L^1)$$

Where $I_n(x)$ is the dyadic interval of the form $[k2^{-n},(k+1)2^{-n})$ containing x and $|I_n(x)| = 2^{-n}$ is the length of $I_n(x)$. The dyadic Hardy space H^p $(1<p<\infty)$ is the collection of function $f\in L^1$ such that

$$\|f\|_{H^p} = \|f^*\|_p < \infty\,.$$

It is clear that $\|f\|_1 \le \|f\|_{H^1}$ and H^1 is a complete proper subspace of L^1.

It is known (see [7]) that for $p > 1$

$$\|f\|_p \le \|f^*\|_p \le \frac{p}{p-1}\|f\|_p$$

and consequently $H^p = L^p \; if \; 1 < p < \infty$ and in this case the L^p-norm and H^p-norms are equivalent.

The dyadic Hardy space $H = H^1$ has atomic characterization [8]. A function $\beta \in L^\infty$ is called dyadic atom if either $\beta = 1$ or there exists dyadic interval $J \subseteq [0,1)$ such that

(i) $\{\beta \neq 0\} = \{x \in [0,1) \,:\, \beta(x) \neq 0\} \subseteq J$

(ii) $\|\beta\|_\infty = |J|^{-1}$

(iii) $\int_0^1 \beta(t)\,dt = 0$

A function $f \in L^1$ belongs to H if and only if there exist dyadic atom $\beta_0, \beta_1, ---$ and a sequence $\{a_n\,, n \in N\}$ such that

$$f = \sum_{n=0}^{\infty} a_n\beta_n, \quad \|a\|_{l^1} = \sum_{n=0}^{\infty} |a_n| < \infty\,, \tag{1.5}$$

furthermore,

$$\|f\|_H \le \inf \|a\|_{l^1} \le 25\,\|f\|_H\,,$$

where infimum is taken over all sequences $a \in l^1$ such that (1.5) holds for some atoms $\beta_0, \beta_1, ---$.

The classical Hardy space consists of functions f, analytic on unit disc, which satisfy

$$\sup_{0 \le r < 1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|F(re^{i\theta})\right|^p\right)^{1/p} < \infty.$$

By taking real part of the boundary functions

$$\lim_{r\uparrow 1} F\left(re^{i\theta}\right)(0 < \theta < 2\pi)$$

and identifying the boundary of the unit disc with the interval [-1,1), one generates the classical non periodic Hardy space S^p on [0,1) , $(1 \le p < \infty)$. As in the dyadic case S^p and L^p are isomorphic for $1 < p < \infty$, and $S = S^1$ is a proper Subspace of L^1. Moreover S has an atomic characterization just like that of H. The essential difference is that an atom for S can be supported on non dyadic intervals. Thus every dyadic atom for H can be supported on non dyadic intervals. Thus every dyadic atom is an atom for S but not conversely. Hence $H \subseteq S$

To obtain conditions on the series $\sum\limits_{k=0}^{\infty} a_k u_k$ sufficient to L^1 - convergence. We introduce a class of sequences induced by norm $\|.\|_X$. For a sequence $b = \{b_n, n \in P\}$, set

$$B^n = \sum_{k=2^{n-1}}^{2^n-1} a_k \chi \left[k2^{-n}, (k+1)\, 2^{-n}\right) , \quad (n \in P),$$

and introduce the sequence norm

$$\|b\|_X = \sum_{n=0}^{\infty} 2^n \,\| \,|B\,|^n \|_X \,.$$

If $X = L^p$ we obtain the class of sequences introduced by Fomin [3].

Obviously

$$\|b\|_{L^1} \leq \|b\|_H \leq \|b\|_{L^p} \quad (1 < p \leq \infty)$$

A summation by parts yields that the nth partial sum of the above series can be written in the form

$$S_n = \sum_{k=0}^{n-1} a_k u_k = \sum_{k=1}^{n} \Delta a_k D_k + a_n D_n \quad (n \in P)$$

where $\Delta a_k = a_{k-1} - a_k. \quad (k \in P)$.

The convergence of wavelets has been investigated by many authors. Y Meyer [6.ch.2] was among the first to study the convergence of wavelet expansions. He showed that if the mother wavelet is r- regular, the orthogonal wavelet expansion of a function will converge to it in the sense of $L^p(\Re)$, $1 \leq p < \infty$, and in the sense of some Sobolov spaces as well. G.Walter [10] proved that the orthogonal wavelet expansion of a function $f \in L^1 \bigcap L^2$ converges to f pointwise at every point of continuity of f and uniformly on compact subsets of any interval (a, b) on which f is continuous. Later, he relaxed this condition and assumed that the scaling function ϕ satisfies the condition $|\phi(x)| \leq \frac{C}{(1+|x|)^3}$. In [4, 5], S. Kelly, M. Kon, and L. Raphael improved Walter's result by proving pointwise convergence of orthogonal wavelet expansions not only under less stringent conditions, but also by extending them to n-dimensions.

2. Definitions and Notations

Wavelet analysis of a function $f \in L^2(\Re)$ basically consists in the decomposition of 'f' as a sum of wavelets

$$\psi_{b.a} = |a|^{\frac{-1}{2}} \,\psi \left(\frac{x-b}{a}\right),$$

where $\psi_{b,a}$ are dilated and translated copies of mother wavelet ψ and these functions are scaled so that their $L^2(\Re)$ norms are independent of 'a'.

For any $j, k \in Z$, we have

$$\left\| f\left(2^{j}. - k\right) \right\|_2 = [\int_{-\infty}^{\infty} \left| f\left(2^{j}x - k\right) \right|^2 dx]^{1/2} = 2^{-j/2}$$

Hence if a function $\psi \in L^2(\Re)$ has a unit length, then all the functions $\psi_{j,k}$ defined by

$$\psi_{j,k}(x) = 2^{j/2}\, \psi\left(2^{j}x - k\right), \quad j, k \in Z, \tag{2.1}$$

also have a unit length i.e.,

$$\|\psi_{j,k}\|_2 = \|\psi\|_2 = 1\,, j, k \in Z\,.$$

A function $\psi \in L^2(\Re)$ is called an orthogonal wavelet (o.n wavelet), if the family $\{\psi_{j.k}\}$, as defined in (2.1) is an orthonormal basis of $L^2(\Re)$, i.e.,

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\, \delta_{k,m}\,, \; j, k, l, m \in Z$$

where

$$\delta_{j,k} = \begin{cases} 0\;, & when\ j \neq k \\ 1\;, & when\ j = k \end{cases}$$

is the Kronecker delta defined on $Z \times Z$. Moreover, every $f \in L^2(\Re)$ can be written as

$$f(x) = .\sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x) \tag{2.2}$$

This series representation of ‘f’ is called wavelet series. Analogous to the notion of Fourier coefficients the wavelet coefficient $c_{j,k}$ are given by

$$c_{j,k} = \langle f, \psi_{j,k} \rangle = \int_{-\infty}^{\infty} f(x)\, \overline{\psi_{j,k}(x)}\, dx\,. \tag{2.3}$$

That is, if we define an integral wavelet transform W_ψ on $L^2(\Re)$ by

$$(W_\psi f)\,(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x)\, \psi\left(\overline{\frac{x-b}{a}}\right) dx\,, \; f \in L^2(\Re)$$

The wavelet coefficient in (2.2) and (2.3) become

$$c_{j,k} = (W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right).$$

Denote $\psi = \{\,\psi_{j,k},\, j, k \in Z\}$ an orthonormal wavelet system defined on $I = [0, 1]$. The Dirchlet kernels with respect to orthonormal wavelet system ψ are denoted by

$$D_{m,n} = \sum_{\substack{|j| < m \\ |k| < n}} \psi_{j,k}, (m, n \in P = N/0)$$

Let us also denote

$$M_{m,n}(c,\psi) = \frac{1}{mn}\int_I \left| \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} c_{j,k} D_{j,k}(x) \right| dx$$

where $c = (c_{j,k})$ are wavelet coefficients of the wavelet series,

$$\sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}. \tag{2.4}$$

The partial sum of (2.4) is written as

$$S_{p,q} = \sum_{j=-p}^{q}\sum_{k=-p}^{q} c_{j,k}\psi_{j,k}.$$

Set

$$B^{m,n} = \sum_{j=2^{m-1}}^{2^m-1}\sum_{k=2^{n-1}}^{2^n-1} c_{j,k}\chi[j2^{-m},(j+1)2^{-m}]\chi[k2^{-n},(k+1)2^{-n}],$$

and introduce a norm

$$\|c\|_X = \sum_{m,n=0}^{\infty} 2^{m+n}\||B^{m+n}|\|_X.$$

3. Main Results

In this section we will prove Sidon type inequalities for wavelets and convergence of wavelet series in L^1- norm.

Theorem 3.1. *Suppose that for wavelet system $\psi = \{\psi_{j,k}\, j,k \in Z\}$*

$$M_{m,n}(c,\psi) = \frac{1}{mn}\int_I \left| \sum_{j=0}^{m-1}\sum_{k=0}^{n-1} c_{j,k} D_{j,k}(x) \right| dx$$

$$M(c,\psi) = \sup_{m,n} M_{m,n}(c,\psi) \le C\,\|c\|_\infty \tag{3.1}$$

is satisfied with constant $C > 0$ independent of wavelet coefficients $c = \{c_{j,k}\}$

(i) *If for every $m,n,i,t \in N$, $|\psi_{m,n}| = 1$, and*

$$D_{(m+i,n+t)} - D_{(m,n)} = \psi_{m,n} D_{i.t}\,, \tag{3.2}$$

then

$$M(c,\psi) \le C_1\ \|c\|_S\,, \tag{3.3}$$

for all $c = \{c_{j,k}\}$, where $C_1 > 0$ is constant independent of c.

(ii) *If for every $l, r, s \in N$ and $0 \leq j, k < 2^s$, $\left|\psi_{(j,k)2^s}\right| = 1$, where $\psi_{(j,k)2s} = \psi_{j2^s,k2^s}$ and*

$$D_{(l,r)2^s+(j,k)} - D_{(l,r)2^s} = \psi_{(l,r)2^s} D_{j,k}, \tag{3.4}$$

then

$$M\,(c, \psi) \leq C_2\, \|c\|_H \tag{3.5}$$

for all $c = \{c_{j,k}\}$, where $C_2 > 0$ is constant independent of c, and

$$D_{(l,r)2^s+(j,k)} = D_{(l2^s+j,r2^s+k)}$$

Theorem 3.2. *Let ψ be a wavelet system. If*

$$\|D_{p,q}\| = 0\,(\log pq) \text{ as } p, q \to \infty, \tag{3.6}$$

then the wavelet series converges in L^1- norm.

Proof of Theorem 3.1: First we show that for every $m, n \in N$ and every $c = \{c_{j,k}\}$

$$M_{m,n}\,(c, \psi) \leq C_1\, \|A_{m,n}\|_{\mathrm{S}}\,. \tag{3.7}$$

where $A_{m,n} = \sum\limits_{j=0}^{m-1} \sum\limits_{k=0}^{n-1} c_{j,k} \chi[j, (j+1))\, \chi[k, (k+1))$.

To prove (3.7) let the sequence c of wavelet coefficients $c = \{c_{j,k}\}$ be such that the corresponding function $A_{m,n}$ is a atom supported in $J = [l, (l+1))$ and $k = [r, (r+1))$. Set $m = l$ and $n = r$. In this case

(a) $c_{j,k} = 0 \left(\begin{array}{l} j \notin [l, (l+1)) \\ k \notin [r, (r+1)) \end{array} \right.$

(b) $\sum\limits_{j=o}^{l-1} \sum\limits_{k=0}^{r-1} c_{j,k} = 0$

(c) $|c_{j,k}| \leq l\,r\,, \quad 0 \leq j < m, \;\; 0 \leq k < n\,.$

Thus

$$M_{m,n}\,(c, \psi) = \frac{1}{m\,n} \int\limits_I \left| \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} c_{j,k} D_{j,k}\,(t) \right| dt$$

$$= \frac{1}{m\,n} \int\limits_I \left| \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} c_{(l,r)+(j,k)} \left(D_{(l,r)+(j,k)}\,(t)\right) - D_{(l,r)}\,(t) \right| dt$$

From (3.2) we get, $= \frac{1}{m\,n} \int\limits_I \left| \sum\limits_{j=0}^{m-1} \sum\limits_{k=0}^{n-1} c_{(l,r)+(j,k)} D_{j,k}\,(t) \right| dt$

Since by (3.1) $\int\limits_I \left| \sum\limits_{j=0}^{m-1} \sum\limits_{k=0}^{n-1} c_{(l,r)+(j,k)} D_{j,k}\,(t) \right| dt \leq \max\limits_{\substack{l < i\,(r+1) \\ r < m < (r+1)}} |c_{i,m}|$

By (c) we get

$$M_{m,n}\,(c, \psi) \leq \frac{1}{m\,n}\, l\,r = 1$$

Replacing atomic decomposition by dyadic atomic decomposition (ii) can be proved in a similar way.

Remark: For every $c = \{c_{j,k}\}$ and $1 < p \leq \infty$, then

$$M(c,\psi) \leq C\|c\|_H \leq C\frac{p}{p-1}\|c\|_p,$$

where $C > 0$ is an absolute constant? □

Proof of Theorem 3.2: If we choose $p < l$ and $q < r$ where $2^{m-1} < \ l, r \ < 2^m$ and $2^{n-1} < \ p, q \ < 2^n$. Then

$$\|S_{p,q} - S_{l-1,r-1}\|_1 \leq \left\|\sum_{i=l}^{p}\sum_{j=r}^{q}\Delta\, b_{i,j}D_{i,j}\right\|_1 + |c_{l-1,r-1}| \ \|D_{l-1,r-1}\|_1 + |c_{p,q}| \ \|D_{p,q}\|_1$$

where $\Delta\, b_{i,j} = b_{i\,,j} - b_{i+1,j} - b_{i,j+1} + b_{i+1,j+1}$. On the basis of $\|D_{p,q}\| \ = 0$ (log $p\,q$) as $p, q \to \infty$, the second and third term tends to zero as $p, q, l, r \to \infty$. On the basis of above remark the first term is estimated and is equivalent to

$$\left\|\sum_{i=l}^{p}\sum_{j=r}^{q}\Delta\, b_{i,j}D_{i,j}\right\|_1 \leq 2^{m+n}\,\|B^{m,n}\|_1 + \sum_{I_1=m+1}^{\infty}\ \sum_{I_2=n+1}^{\infty} 2^{(I_1+I_2)}\ \|\,|B^{I_1,I_2}|\,\|_H + 2^{m+n}\,\|\,|B^{m,n}|\,\|_H$$

which tends to zero as $p, q, l, r \to \infty$. □

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Almansi Decomposition for Dunkl-Helmholtz Operators

Guangbin Ren and Helmuth R. Malonek

Abstract. We consider the iterated Dunkl-Helmholtz equation $(\Delta_h - \lambda)^n f = 0$ for nonzero λ in a domain of $\mathbb{R}^N$. Here $\Delta_h = \sum_{j=1}^N \mathcal{D}_j^2$ is the Dunkl Laplacian, and $\mathcal{D}_j$ is the Dunkl operator attached to the Coxeter group G associated with the reduced root system R,

$$\mathcal{D}_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(\sigma_v x)}{\langle x, v\rangle} v_j,$$

where κ_v is a multiplicity function on R and σ_v is the reflection with respect to the root v.

We prove that any solution f of the iterated Dunkl-Helmholtz equation has a decomposition of the form

$$f(x) = f_0(x) + R_\mu f_1(x) + \cdots + R_\mu^{n-1} f_{n-1}(x), \qquad \forall\ x \in \Omega,$$

where f_j are annihilated by $\Delta_h - \lambda$, μ is a fixed but arbitrary complex number, and $R_\mu^n = (R_\mu)^n$ are given by $R_\mu = \mu I + R_0$, with I the identity operator and R_0 the Euler operator.

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1. Introduction

Polyharmonic theory is a powerful tool in many fields. As extension of polynomials, polyharmonic functions play a key role in multivariate approximation [1]. The polyharmonic theory has its roots in the theory of elasticity [2] and in radar imaging [3]. The fundamental result in the theory of polyharmonic functions is the

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celebrated Almansi theorem [4, 5], which shows that for any polyharmonic function f of degree n in a starlike domain Ω in $\mathbb{R}^N$ with respect to 0, i.e., $\Delta^n f = 0$, there exist unique harmonic functions $f_0, \ldots, f_{n-1}$ such that

$$f(x) = f_0(x) + |x|^2 f_1(x) + \cdots + |x|^{2(n-1)} f_{n-1}(x), \qquad \forall\, x \in \Omega.$$

We refer to [6] for the applications of the Almansi decomposition in partial differential equations.

As same as the iterated Laplacian, the iterated Helmholtz operators $(\Delta - \lambda)^n$, $\lambda \in \mathbb{C} \setminus \{0\}$, have also gained considerable interest in mathematical physics [7]. These operators can be further considered in the setting of the Dunkl theory. The Dunkl theory is an extension of the classical harmonic theory. In 1989, Dunkl [8] constructed for each Coxeter group a family of commutative differential-difference operators $\mathcal{D}_j$, called Dunkl operators, which can be considered as perturbations of the usual partial derivatives by reflection parts. These operators stem from the analysis of quantum many body system of Calogero-Moser-Sutherland type [9] in mathematical physics. They also have their root in the theory of special functions [10]. With Dunkl operators substituted for partial derivatives $\frac{\partial}{\partial x_j}$, one can define the Laplacian in the Dunkl setting, which is a parametrized operator and invariant under reflection groups. These parametrized Laplacian underlies the Dunkl theory [11]. The Helmholtz operators in the setting of the Dunkl theory are called the Dunkl-Helmholtz operators.

The purpose of this article is to extend Almansi's theorem from Laplacian to the Dunkl-Helmholtz operators. A new phenomena shall arise that the auxiliary function jumps from $|x|^2$ to the Euler operator as λ being away from zero in the Almansi decompositions for $\Delta - \lambda$.

2. Main Theorem

For a nonzero vector $v \in \mathbb{R}^N$, the reflection σ_v in the hyperplane orthogonal to v is defined by

$$\sigma_v x := x - 2\frac{\langle x, v\rangle}{|v|^2} v, \qquad x \in \mathbb{R}^N,$$

where the symbol $\langle x, y\rangle$ denotes the usual Euclidean inner product and $|x|^2 = \langle x, x\rangle$.

A root system R is a finite set of nonzero vectors in $\mathbb{R}^N$ such that $\sigma_v R = R$ and $R \cap \mathbb{R}v = \{\pm v\}$ for all $v \in R$.

The Coxeter group G (or the finite reflection group) generated by the root system R is the subgroup of the orthogonal group $O(N)$ generated by $\{\sigma_u : u \in R\}$.

The positive subsystem R_+ is a subset of R such that $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin.

A multiplicity function κ_v is a G-invariant complex valued function defined on R, i.e, $\kappa_v = \kappa_{gv}$ for all $g \in G$.

The Dunkl operators $\mathcal{D}_j$, associated with the Coxeter group G and the multiplicity function κ, are the first order differential-difference operator

$$\mathcal{D}_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(\sigma_v x)}{\langle x, v \rangle} v_j.$$

The remarkable property of Dunkl operators is commutativity

$$\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i.$$

The Dunkl Laplacian is defined as

$$\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_N^2.$$

Let Ω be a G-invariant convex domain in $\mathbb{R}^N$ including 0, i.e, $G(\Omega) \subset \Omega$, $0 \in \Omega$, and $tx + (1-t)y \in \Omega$ for all $t \in [0,1]$ and $x, y \in \Omega$. A function $f : \Omega \longrightarrow \mathbb{C}$ in $C^{2n}(\Omega)$ is *Dunkl polyharmonic* of degree n if $(\Delta_h)^n f = 0$.

In [12], the Almansi decomposition theorem is established for Dunkl operators:

Theorem 2.1. *Assume that*

$$\operatorname{Re} \sum_{v \in R_+} \kappa_v > -N/2. \tag{2.1}$$

Let Ω be a G-invariant convex domain in $\mathbb{R}^N$ including 0, where G is the Coxeter group in $\mathbb{R}^N$. If f is a Dunkl polyharmonic function in Ω of degree n, then there exist unique Dunkl harmonic functions $f_0, \ldots, f_{n-1}$ such that

$$f(x) = f_0(x) + |x|^2 f_1(x) + \cdots + |x|^{2(n-1)} f_{n-1}(x), \qquad \forall\, x \in \Omega. \tag{2.2}$$

Conversely, the sum in (2.2)*, with $f_0, \ldots, f_{n-1}$ Dunkl harmonic in Ω, defines a Dunkl polyharmonic function in Ω of degree n.*

In this article we consider the decomposition theorem for the iterated Dunkl-Helmholtz operator.

Definition 2.2. The Dunkl-Helmholtz operator is given by

$$\Delta_{h,\lambda} := \Delta_h - \lambda$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$.

When $\kappa_v = 0$, the Dunkl-Helmholtz operator is just the Helmholtz operator.

We fix $\mu \in \mathbb{C}$ throughout the paper. Let I be the identity operator. For any smooth function f in Ω, we denote the Euler operator

$$R_0 f(x) = \sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j}(x), \qquad x \in \Omega,$$

and denote

$$R_\mu = \mu I + R_0.$$

For simplicity, we write the iterated operators

$$\Delta_{h,\lambda}^n := (\Delta_{h,\lambda})^n, \qquad R_\mu^n := (R_\mu)^n.$$

For $\lambda \in \mathbb{C} \setminus \{0\}$, we denote

$$c_k = c_{k,\lambda} := \frac{1}{k!(2\lambda)^k}.$$

Our main result is contained in the following theorem.

Theorem 2.3. *Let $\kappa : R \to \mathbb{C}$ be an arbitrary multiplicity function on R, Ω be a G-invariant domain in $\mathbb{R}^N$, $\lambda \in \mathbb{C} \setminus \{0\}$, and $\mu \in \mathbb{C}$. If $f \in C^{2n}(\Omega)$ satisfies*

$$\Delta_{h,\lambda}^n f = 0. \tag{2.3}$$

for some positive integer n, then there exist unique functions $f_0, \ldots, f_{n-1}$ annihilated by the Dunkl-Helmholtz operator $\Delta_{h,\lambda}$ such that

$$f(x) = f_0(x) + R_\mu f_1(x) + \cdots + R_\mu^{n-1} f_{n-1}(x), \qquad \forall\, x \in \Omega. \tag{2.4}$$

Moreover the functions $f_0, \ldots, f_{n-1}$ are given by the following formulae:

$$\begin{array}{lcr}
f_0 & = & (I - c_1 R_\mu \Delta_{h,\lambda})(I - c_2 R_\mu^2 \Delta_{h,\lambda}^2) \cdots (I - c_{n-1} R_\mu^{n-1} \Delta_{h,\lambda}^{n-1}) f(x) \\
f_1 & = & c_1 \Delta_{h,\lambda}(I - c_2 R_\mu^2 \Delta_{h,\lambda}^2) \cdots (I - c_{n-1} R_\mu^{n-1} \Delta_{h,\lambda}^{n-1}) f(x) \\
 & \vdots & \\
f_{n-2} & = & c_{n-2} \Delta_{h,\lambda}^{n-2}(I - c_{n-1} R_\mu^{n-1} \Delta_{h,\lambda}^{n-1}) f(x) \\
f_{n-1} & = & c_{n-1} \Delta_{h,\lambda}^{n-1} f(x).
\end{array}$$

Conversely, the sum in (2.4)*, with $f_0, \ldots, f_{n-1}$ annihilated by the Dunkl-Helmholtz operator $\Delta_{h,\lambda}$ in Ω with nonzero λ, defines a function f in Ω satisfying* (2.3)*.*

Remark 2.4. Comparing the Almansi decompositions for the Dunkl operator and the Dunkl-Helmholtz operator, we find the auxiliary function $|x|^2$ in (2.2) changed to R_μ in (2.4) which causes a big difference. The Almansi decomposition for the Dunkl-Helmholtz operator puts no restriction to the multiplicity function κ, in contrast to the case for Dunkl operators. In the literature, it is generally assumed that $\kappa_v \geq 0$ (see [11]).

3. Some Lemmas

We first recall some basic facts in the theory of Dunkl harmonics; see [11].

Let R be a root system in $\mathbb{R}^N$ and G the associated Coxeter group. Fix a positive subsystem R_+ of R. Let $\kappa : R \to \mathbb{C}$ be a fixed multiplicity function $v \mapsto \kappa_v$ on R. The Dunkl operators $\mathcal{D}_j$, associated with the Coxeter group G and the multiplicity function κ, are the first order differential-difference operator. They enjoy the regularity property: If $f \in C^m(\Omega)$ with $m \geq 1$, then $\mathcal{D}_i f \in C^{m-1}(\Omega)$. This follows immediately from the formula

$$\frac{f(x) - f(\sigma_v x)}{\langle x, v \rangle} = \int_0^1 \langle \nabla f(t\sigma_v x + (1-t)x), \frac{2v}{|v|^2} \rangle dt \tag{3.1}$$

for $f \in C^1(\Omega)$ and $v \in R$. Here ∇ is the usual gradient operator.

The Dunkl Laplacian $\Delta_h = \sum_{j=1}^{N} \mathcal{D}_j^2$ can be written as (see [11]),

$$\Delta_h f(x) = \Delta f(x) + 2 \sum_{v \in R_+} \kappa_v \frac{\langle \nabla f(x), v \rangle}{\langle x, v \rangle} - 2 \sum_{v \in R_+} \kappa_v \frac{f(x) - f(\sigma_v x)}{\langle x, v \rangle^2} |v|^2.$$

Here Δ is the ordinary Laplacian operator. When $\kappa = 0$, the Dunkl Laplacian Δ_h is just the Laplacian Δ.

By the regularity of Dunkl operators, Δ_h is a regular operator in any G-invariant convex domain. Namely, if $f \in C^m(\Omega)$ with $m \geq 2$, then $\Delta_h f \in C^{m-2}(\Omega)$.

Consider the natural action of $O(N)$ on functions $f : \mathbb{R}^N \to \mathbb{C}$, given by $gf(x) = f(g^{-1}x)$. The Dunkl Laplacian Δ_h is G-invariant, i.e.,

$$g \circ \Delta_h = \Delta_h \circ g, \qquad \forall\, g \in G.$$

Example. In the one-dimensional case $N = 1$, the root system R is of type A_1, the reflection group $G = \mathbb{Z}_2$, and the multiplicity function is given by a single parameter $\kappa \in \mathbb{C}$. The Dunkl operator $\mathcal{D} := \mathcal{D}_1$ and the Dunkl-Helmholtz Laplacian $\Delta_{h,\lambda}$ are given respectively by

$$\begin{aligned} \mathcal{D} f(x) &= f'(x) + \kappa \frac{f(x) - f(-x)}{x}, \\ \Delta_{h,\lambda} f(x) &= f''(x) + 2\kappa \frac{f'(x)}{x} - 2\kappa \frac{f(x) - f(-x)}{x^2} - \lambda f(x). \end{aligned}$$

If f is an even function, then the third term in the formula of $\Delta_{h,\lambda} f$ vanishes, while the sum of the first two items provides a singular Sturm-Liouville operator.

With the regularity of Dunkl operator in hand, we can establish some lemmas. Recall our assumption that Ω is a G-invariant domain in $\mathbb{R}^N$, $\lambda \in \mathbb{C} \setminus \{0\}$, and $\mu \in \mathbb{C}$.

Let $s \in \mathbb{C}$ such that Re $s > 0$, and denote

$$I_s f(x) = \int_0^1 f(tx) t^{s-1} dt.$$

If Ω is a G-invariant domain in $\mathbb{R}^N$ and Re $s > 0$, then as operators on $C^2(\Omega)$ (see [12])

$$\begin{aligned} \Delta_h I_s &= I_{s+2} \Delta, \\ R_s I_s &= I_s R_s = I. \end{aligned}$$

Lemma 3.1. *If $f \in C^2(\Omega)$ and $\mu \in \mathbb{C}$, then*

$$R_{\mu+2} \Delta_h f = \Delta_h R_\mu f. \tag{3.2}$$

Proof. If Re $s > 0$, then $R_{s+2}\Delta_h = R_{s+2}\Delta_h I_s R_s = R_{s+2} I_{s+2} \Delta_h R_s = \Delta_h R_s$, so that $R_{\mu+2}\Delta_h = (R_{s+2} + (\mu - s)I)\Delta_h = \Delta_h R_s + \Delta_h((\mu - s)I) = \Delta_h R_\mu$. □

As a result, we find that if f is Dunkl harmonic, then so is $R_\mu f$.

Lemma 3.2. *Let $f \in C^2(\Omega)$ be such that $\Delta_{h,\lambda} f = 0$. Then for any $\mu \in \mathbb{C}$ and $n \in \mathbb{N}$*

$$c_n \Delta^n_{h,\lambda} R^n_\mu f = f. \tag{3.3}$$

Proof. We prove the identity by induction on n. As $\Delta_h R_0 = (2I + R_0)\Delta_h$, a simple calculation shows that

$$\Delta_{h,\lambda} R_\mu = R_\mu \Delta_{h,\lambda} + 2\Delta_{h,\lambda} + 2\lambda I. \tag{3.4}$$

The assertion (3.3) with $n = 1$ follows from (3.4). Now we assume the validity of (3.3) for the case $n = r$ and prove the case $n = r + 1$. Let $f \in \ker \Delta_{h,\lambda}$, then by assumption

$$\Delta^{r+1}_{h,\lambda} R^r_\mu f = 0. \tag{3.5}$$

Therefore from (3.4)

$$\Delta^{r+1}_{h,\lambda} R^{r+1}_\mu f = \Delta^r_{h,\lambda}(R_\mu \Delta_{h,\lambda} + 2\Delta_{h,\lambda} + 2\lambda) R^r_\mu f.$$

Denote $C_n = 1/c_n = n!2^n$. Applying (3.5) and the induction hypothesis, we have

$$\Delta^{r+1}_{h,\lambda} R^{r+1}_\mu f = \Delta^r_{h,\lambda} R_\mu \Delta_{h,\lambda} R^r_\mu f + 2\lambda C_r f.$$

It is seen that an additional factor appears after having interchanged R_μ with $\Delta_{h,\lambda}$. Continue in this way we obtain

$$\begin{aligned} \Delta^{r+1}_{h,\lambda} R^{r+1}_\mu f &= \Delta^{r-1}_{h,\lambda} R_\mu \Delta^2_{h,\lambda} R^r_\mu f + 4\lambda C_r f \\ &= \cdots \\ &= R_\mu \Delta^{r+1}_{h,\lambda} R^r_\mu + 2(r+1)\lambda C_r f. \end{aligned}$$

From (3.5), we see that (3.3) holds for $n = r + 1$. This completes the proof. □

4. Proof of the Main Theorem

Now we come to the proof of our main theorem.

Proof of Theorem 2.3. Let $n \in \mathbb{N}$ and denote $H_n = \{f \in C^{2n}(\Omega) : \Delta^n_{h,\lambda} f = 0\}$. For shorthand, we write

$$T_n = (R_\mu)^n.$$

It is sufficient to show that

$$H_n = H_{n-1} + T_{n-1} H_1, \quad n \in \mathbb{N}.$$

Notice that Lemma 3.2 states that in H_1

$$c_n \Delta^n_{h,\lambda} T_n = I. \tag{4.1}$$

We split the proof into two parts.

(i) $H_n \supset H_{n-1} + T_{n-1} H_1$.

As $H_{n-1} \subset H_n$, we only need to show that $T_{n-1} H_1 \subset H_n$. For any $g \in H_1$, it follows from (4.1) that

$$\Delta^n_{h,\lambda}(T_{n-1} g) = c^{-1}_{n-1} \Delta_{h,\lambda}(c_{n-1} \Delta^{n-1}_{h,\lambda} T_{n-1}) g = c^{-1}_{n-1} \Delta_{h,\lambda} g = 0.$$

(ii) $H_n \subset H_{n-1} + T_{n-1}H_1$.

For any $f \in H_n$, we have the decomposition

$$f = (I - c_{n-1}T_{n-1}\Delta_{h,\lambda}^{n-1})f + T_{n-1}(c_{n-1}\Delta_{h,\lambda}^{n-1}f).$$

It is evident that the function between brackets in the second term belongs to H_1. We only need to show that also the first term is in H_{n-1}. This can be verified directly. Indeed

$$\begin{aligned}\Delta_{h,\lambda}^{n-1}(I - c_{n-1}T_{n-1}\Delta_{h,\lambda}^{n-1})f &= (\Delta_{h,\lambda}^{n-1} - (c_{n-1}\Delta_{h,\lambda}^{n-1}T_{n-1})\Delta_{h,\lambda}^{n-1})f \\ &= (\Delta_{h,\lambda}^{n-1} - \Delta_{h,\lambda}^{n-1})f = 0,\end{aligned}$$

as desired.

This proves that $H_n = H_{n-1} + T_{n-1}H_1$. By induction, we can easily deduce that $H_n = H_1 + T_1H_1 + \cdots + T_{n-1}H_1$.

Next we prove that for any $f \in H_n$ the decomposition

$$f = g + T_{n-1}f_{n-1}, \quad g \in H_{n-1},\ f_{n-1} \in H_1$$

is unique. In fact, for such a decomposition, applying $\Delta_{h,\lambda}^{n-1}$ on both sides we obtain

$$\begin{aligned}\Delta_{h,\lambda}^{n-1}f &= \Delta_{h,\lambda}^{n-1}g + \Delta_{h,\lambda}^{n-1}T_{n-1}f_{n-1} \\ &= c_{n-1}^{-1}f_{n-1}.\end{aligned}$$

Therefore

$$f_{n-1} = c_{n-1}\Delta_{h,\lambda}^{n-1}f,$$

so that

$$g = f - T_{n-1}f_{n-1} = (I - c_{n-1}T_{n-1}\Delta_{h,\lambda}^{n-1})f.$$

Thus the uniqueness follows by induction.

To prove the converse, we see from (3.3) that, for any $n \in \mathbb{N}$, $\Delta_{h,\lambda}^{n+1}R_\mu^n H_1 = 0$. Replacing n by j, we have

$$\Delta_{h,\lambda}^{n}R_\mu^j H_1 = 0$$

for any $n > j$. So $R_\mu^j H_1 \subset H_n$ for any $j < n$, as desired. □

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An Uncertainty Principle for Operators

Michael G. Cowling and M. Sundari

Abstract. Hardy's Uncertainty Principle asserts that if f is a function on $\mathbb{R}^n$ such that $\exp(\alpha|\cdot|^2)f$ and $\exp(\beta|\cdot|^2)\hat{f}$ are bounded, where $\alpha\beta > \frac{1}{4}$, then $f = 0$. In this paper, we prove a version of Hardy's result for operators.

Perhaps the basic difficulty in Fourier analysis, which is evident in the theory of wavelets, is that nontrivial functions cannot be localised in time and frequency. One manifestation of this is Hardy's Uncertainty Principle: suppose that f is a function on $\mathbb{R}^n$ and that

$$|f(x)| \le C \exp(-\alpha|x|^2) \qquad \forall x \in \mathbb{R}^n$$
$$|\hat{f}(\xi)| \le C \exp(-\beta|\xi|^2) \qquad \forall \xi \in \mathbb{R}^n.$$

If moreover $\alpha\beta > 1/4$, then $f = 0$; if $\alpha\beta = 1/4$, then $f(x) = \exp(-\alpha|x|^2)$, up to a constant. In this theorem, and in the rest of this paper, the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \exp(-i\xi \cdot x)\, dx \qquad \forall \xi \in \mathbb{R}^n.$$

A proof of Hardy's Uncertainty Principle (with a different definition of the Fourier transform and therefore stated slightly differently) may be found in [2]. In this paper, we prove a version of Hardy's result for operators.

1. The Main Result

Suppose that K is an operator on $L^2(\mathbb{R}^n)$ which is given by a kernel k, that is,

$$Kf(x) = \int_{\mathbb{R}^n} k(x, y)\, f(y)\, dy \qquad \forall x \in \mathbb{R}^n$$

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for all f in $L^2(\mathbb{R}^n)$. We call K a kernel operator. Kernel operators may be given different partial orderings. We write $|K_1| \leq |K_2|$ if

$$\|K_1 f\|_2 \leq \|K_2 f\|_2 \qquad \forall f \in L^2(\mathbb{R}^n)$$

and $|k_1| \leq |k_2|$ if

$$|k_1(x,y)| \leq |k_2(x,y)| \qquad \forall x, y \in \mathbb{R}^n.$$

We may omit some of the absolute value signs if one of the operators is positive, or if one of the kernels is positive.

The heat semigroup provides an important family of operators on $L^2(\mathbb{R}^n)$. We define the heat kernel p_s by

$$p_s(x,y) = C(s) \exp\Big(-\frac{|x-y|^2}{4s}\Big) \qquad \forall x, y \in \mathbb{R}^n$$

where $C(s)$ depends on n and on s but its precise value is irrelevant, and define the heat operator P_s by

$$P_s f(x) = \int_{\mathbb{R}^n} p_s(x,y)\, f(y)\, dy \qquad \forall x \in \mathbb{R}^n$$

for all f in $L^2(\mathbb{R}^n)$. Often P_s is written as $\exp(s\Delta)$ or $\exp(-s\Delta)$, depending on how the Laplacian is normalised.

Here is our version of Hardy's theorem for operators. If K is a convolution operator, this boils down to the classical version of Hardy's Uncertainty Principle, but otherwise it is more general.

Theorem 1. *Suppose that K is the operator on $L^2(\mathbb{R}^n)$ associated to the kernel k, and that*

$$\begin{aligned} |k| &\leq C p_s \\ |K| &\leq C P_t \\ s &< t. \end{aligned}$$

Then $K = 0$.

Before we proceed to the proof, we note that if m is any bounded measurable function on $\mathbb{R}^n$ and $k(x,y) = m(x)\, p_s(x,y)$, then the first two inequalities of the theorem hold when $t = s$; but clearly k is more general than a Gaussian.

Proof. Take any smooth compactly supported function b on $\mathbb{R}^n$. We will show that $K^* b = 0$. Since b is arbitrary, this establishes that $K^* = 0$, and hence $K = 0$.

To do this, take s' such that $s < s' < t$. Then

$$\begin{aligned} |K^* b(x)| &= \Big| \int_{\mathbb{R}^n} \bar{k}(y,x)\, b(y)\, dy \Big| \\ &\leq C(s) \int_{\mathbb{R}^n} \exp\Big(-\frac{|y-x|^2}{4s}\Big)\, |b(y)|\, dy \\ &\leq C(s,s',b) \exp\Big(-\frac{|x|^2}{4s'}\Big) \qquad \forall x \in \mathbb{R}^n. \end{aligned}$$

On the other hand, for any nonnegative integer k,

$$\begin{aligned}
\|\Delta^k K^* b\|_2 &= \sup\left\{\left|\left\langle \Delta^k K^* b, g\right\rangle\right| : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_2 \le 1\right\} \\
&\le \|b\|_2 \sup\left\{\left\|K\Delta^k g\right\|_2 : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_2 \le 1\right\} \\
&\le C\,\|b\|_2 \sup\left\{\left\|P_t\Delta^k g\right\|_2 : g \in C_c^\infty(\mathbb{R}^n),\ \|g\|_2 \le 1\right\} \\
&= C\,\|b\|_2 \sup\left\{\exp(-t|\xi|^2)\,|\xi|^{2k} : \xi \in \mathbb{R}^n\right\} \\
&= C\left(\frac{k}{et}\right)^k \|b\|_2
\end{aligned}$$

(when $k = 0$, the right hand side is to be interpreted as $C\|b\|_2$). Now take t' such that $s' < t' < t$. As noted by Thangavelu [4],

$$\begin{aligned}
\|\exp(t'|\cdot|^2)(K^*b)\hat{}\,\|_2 &\le \sum_{k\in\mathbb{N}} \frac{1}{k!}\|(t'|\cdot|^2)^k (K^*b)\hat{}\,\|_2 \\
&\le C \sum_{k\in\mathbb{N}} \frac{(t')^k}{k!}\left(\frac{k}{et}\right)^k \|b\|_2 \\
&= C \sum_{k\in\mathbb{N}} \left(\frac{t'}{t}\right)^k \left(\frac{k}{e}\right)^k \frac{1}{k!}\,\|b\|_2 \\
&< \infty.
\end{aligned}$$

Now a form of Hardy's principle proved by Cowling and Price [1], which also follows from a result of Beurling proved in Hörmander [3], states that if

$$\begin{gathered}
\|\exp\left(\frac{|\cdot|^2}{4s'}\right) f\|_\infty < \infty \\
\|\exp(t'|\cdot|^2)\,\hat{f}\|_2 < \infty \\
s' \le t'
\end{gathered}$$

then $f = 0$. We apply this with K^*b in place of f to deduce the required result. □

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Uncertainty Principle for Clifford Geometric Algebras $Cl_{n,0}\,,\ n = 3\,(\mathrm{mod}\,4)$ Based on Clifford Fourier Transform

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Soli Deo Gloria

Abstract. First, the basic concepts of the multivector functions, vector differential and vector derivative in geometric algebra are introduced. Second, we define a generalized real Fourier transform on Clifford multivector-valued functions ($f : \mathbb{R}^n \to Cl_{n,0},\ n = 3\,(\mathrm{mod}\,4)$). Third, we introduce a set of important properties of the Clifford Fourier transform on $Cl_{n,0},\ n = 3\,(\mathrm{mod}\,4)$ such as differentiation properties, and the Plancherel theorem. Finally, we apply the Clifford Fourier transform properties for proving a *directional* uncertainty principle for $Cl_{n,0},\ n = 3\,(\mathrm{mod}\,4)$ multivector functions.

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Keywords. Vector derivative, multivector-valued function, Clifford (geometric) algebra, Clifford Fourier transform, uncertainty principle.

1. Introduction

In the field of applied mathematics the Fourier transform has developed into an important tool. It is a powerful method for solving partial differential equations. The Fourier transform provides also a technique for signal analysis where the signal from the original domain is transformed to the spectral or frequency domain. In the frequency domain many characteristics of the signal are revealed. With these facts in mind, we extend the Fourier transform in geometric algebra.

Brackx et al. [1] extended the Fourier transform to multivector valued function-distributions in $Cl_{0,n}$ with compact support. They also showed some properties of this generalized Fourier transform. A related applied approach for hypercomplex Clifford Fourier transformations in $Cl_{0,n}$ was followed by Bülow et al. [2].

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By extending the classical trigonometric exponential function $\exp(j\,\boldsymbol{x} * \boldsymbol{\xi})$ (where $*$ denotes the scalar product of $\boldsymbol{x} \in \mathbb{R}^m$ with $\boldsymbol{\xi} \in \mathbb{R}^m$, j the imaginary unit) in [3, 4], McIntosh et al. generalized the classical Fourier transform. Applied to a function of m real variables this generalized Fourier transform is holomorphic in m complex variables and its inverse is *monogenic* in $m+1$ real variables, thereby effectively extending the function of m real variables to a monogenic function of $m+1$ real variables (with values in a *complex* Clifford algebra). This generalization has significant applications to harmonic analysis, especially to singular integrals on surfaces in $\mathbb{R}^{m+1}$. Based on this approach Kou and Qian obtained a Clifford Payley-Wigner theorem and derived Shannon interpolation of band-limitted functions using the monogenic sinc function [5, and references therein]. The Clifford Payley-Wigner theorem also allows to derive left-entire (left-monogenic in the whole $\mathbb{R}^{m+1}$) functions from square integrable functions on $\mathbb{R}^m$ with compact support.

In this paper we adopt and expand[1] to $\mathcal{G}_n,\ n = 3 \,(\mathrm{mod}\, 4)$ the generalization of the Fourier transform in Clifford geometric algebra $\mathcal{G}_3$ recently suggested by Ebling and Scheuermann [7]. We introduce detailed properties of the real[2] Clifford geometric algebra Fourier transform (CFT), which we subsequently use to define and prove a general directional uncertainty principle for $\mathcal{G}_n$ multivector functions.

2. Clifford's Geometric Algebra $\mathcal{G}_n$ of $\mathbb{R}^n$

Let us consider now and in the following an orthonormal vector basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ of the real n-dimensional Euclidean vector space $\mathbb{R}^n$ with $n = 3 \,(\mathrm{mod}\, 4)$. Each basis vector has unit square, i.e. $\boldsymbol{e}_k^2 = 1$, $1 \leq k \leq n$. The geometric algebra over $\mathbb{R}^n$ denoted by $\mathcal{G}_n$ then has the graded 2^n-dimensional basis

$$\{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n, \boldsymbol{e}_{12}, \boldsymbol{e}_{31}, \boldsymbol{e}_{23}, \ldots, i_n = \boldsymbol{e}_1\boldsymbol{e}_2 \ldots \boldsymbol{e}_n\}. \tag{2.1}$$

For the simplest case of $n = 3$ the basis reduces to

$$\begin{aligned} & \{1, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_{23}, \boldsymbol{e}_{31}, \boldsymbol{e}_{12}, i_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3\} \\ = \ & \{1, i_3, \boldsymbol{e}_1, i_3\boldsymbol{e}_1 = \boldsymbol{e}_{23}, \boldsymbol{e}_2, i_3\boldsymbol{e}_2 = \boldsymbol{e}_{31}, \boldsymbol{e}_3 = \boldsymbol{e}_2\boldsymbol{e}_{23}, \boldsymbol{e}_{12} = \boldsymbol{e}_{31}\boldsymbol{e}_{23}\} \qquad (2.2) \\ \doteq \ & \{1, i_3, \boldsymbol{e}_{23}, i_3\boldsymbol{e}_{23} = -\boldsymbol{e}_1, \boldsymbol{e}_{31}, i_3\boldsymbol{e}_{31} = -\boldsymbol{e}_2, \boldsymbol{e}_{12} = \boldsymbol{e}_{31}\boldsymbol{e}_{23}, i_3\boldsymbol{e}_{12} = -\boldsymbol{e}_3\}. \end{aligned}$$

Equation (2.2) exemplifies for $n = 3$ the general isomorphisms

$$\mathcal{G}_n \approx \mathcal{G}_{n-1} \times \mathbb{C} \approx \mathcal{G}_{0,n-1} \times \mathbb{C}, \tag{2.3}$$

[1]For further details and proofs in the case of $n = 3$ compare [6]. In the geometric algebra literature [8] instead of the mathematical notation $Cl_{p,q}$ the notation $\mathcal{G}_{p,q}$ is widely in use. It is convention to abbreviate $\mathcal{G}_{n,0}$ to $\mathcal{G}_n$.

[2]The meaning of *real* in this context is, that we use the n-dimensional volume element $i_n = \boldsymbol{e}_1\boldsymbol{e}_2 \ldots \boldsymbol{e}_n$ of the geometric algebra $\mathcal{G}_n$ over the field of the reals $\mathbb{R}$ to construct the kernel of the Clifford Fourier transformation of definition 4.1. This i_n has a clear geometric interpretation. Note that $i_n^2 = -1$ for $n = 2, 3 \,(\mathrm{mod}\, 4)$.

which can be exploited to transfer results from a complexified Clifford algebra $\mathcal{G}_{0,n-1} \times \mathbb{C}$ to the real geometric algebra $\mathcal{G}_n$.

The *grade selector* is defined as $\langle M\rangle_k$ for the k-vector part of M, especially $\langle M\rangle = \langle M\rangle_0$. Then M can be expressed as

$$M = \langle M\rangle + \langle M\rangle_1 + \langle M\rangle_2 + \ldots + \langle M\rangle_n. \tag{2.4}$$

The *reverse* of M is defined by the anti-automorphism

$$\widetilde{M} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle M\rangle_k. \tag{2.5}$$

The *square norm* of M is defined by

$$\|M\|^2 = \langle M\widetilde{M}\rangle, \tag{2.6}$$

where

$$\langle M\widetilde{N}\rangle = M * \widetilde{N} = \sum_A \alpha_A \beta_A \tag{2.7}$$

is a real valued (inner) *scalar product* for any M, N in $\mathcal{G}_n$ with $M = \sum_A \alpha_A \boldsymbol{e}_A$ and $N = \sum_A \beta_A \boldsymbol{e}_A$, $A \in \{0, 1, 2, \ldots, n, 12, 31, 23, \ldots, 12\ldots n\}$, $\alpha_A, \beta_A \in \mathbb{R}$, and $\boldsymbol{e}_A$ the basis elements of (2.1). Especially for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ we get (using the costumary dot)

$$\boldsymbol{a} \cdot \boldsymbol{b} = \langle \boldsymbol{a}\boldsymbol{b}\rangle = \boldsymbol{a} * \boldsymbol{b} = \sum_{A=1}^{n} \alpha_A \beta_A \tag{2.8}$$

As a consequence we obtain the *multivector Cauchy-Schwarz inequality*

$$|\langle M\widetilde{N}\rangle|^2 \;\leq\; \|M\|^2\, \|N\|^2 \quad \forall\, M, N \in \mathcal{G}_n. \tag{2.9}$$

3. Multivector Functions, Vector Differential and Vector Derivative

Let $f = f(\boldsymbol{x})$ be a multivector-valued function of a vector variable $\boldsymbol{x}$ in $\mathcal{G}_n$. For an arbitrary vector $\boldsymbol{a}$ we define[3] the *vector differential* in the $\boldsymbol{a}$ direction as

$$\boldsymbol{a} \cdot \nabla f(\boldsymbol{x}) = \lim_{\epsilon \to 0} \frac{f(\boldsymbol{x} + \epsilon \boldsymbol{a}) - f(\boldsymbol{x})}{\epsilon} \tag{3.1}$$

provided this limit exists and is well defined. The basis independent linear *vector derivative* ∇ defined in [8, 9] obeys equation (3.1) for all vectors $\boldsymbol{a}$ and can be expanded as

$$\nabla = \boldsymbol{e}_k \partial_k = \boldsymbol{e}_1 \partial_1 + \boldsymbol{e}_2 \partial_2 + \ldots + \boldsymbol{e}_n \partial_n, \tag{3.2}$$

For use in later sections we state a number of elementary properties of the vector differential and the vector derivative (compare [8, 9])

[3] Bracket convention: $A \cdot BC = (A \cdot B)C \neq A \cdot (BC)$ and $A \wedge BC = (A \wedge B)C \neq A \wedge (BC)$ for multivectors $A, B, C \in \mathcal{G}_{p,q}$. The vector variable index $\boldsymbol{x}$ of the vector derivative is dropped: $\boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{\nabla}$ and $\boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} = \boldsymbol{a} \cdot \boldsymbol{\nabla}$, but not when differentiating with respect to a different vector variable (compare e.g. proposition 3.2).

Proposition 3.1 (Chain rule for $g \circ \lambda, \lambda \in \mathbb{R}$). *For $f(\boldsymbol{x}) = g(\lambda(\boldsymbol{x}))$, $\lambda(\boldsymbol{x}) \in \mathbb{R}$,*

$$\boldsymbol{a} \cdot \nabla f = \{\boldsymbol{a} \cdot \nabla \lambda(\boldsymbol{x})\} \frac{\partial g}{\partial \lambda}\,. \tag{3.3}$$

Proposition 3.2 (Derivative from differential).

$$\nabla f = \nabla_{\boldsymbol{a}} \left(\boldsymbol{a} \cdot \nabla f\right). \tag{3.4}$$

Differentiating twice with the vector derivative, we get the differential Laplacian operator ∇^2. We can write $\nabla^2 = \nabla \cdot \nabla + \nabla \wedge \nabla$. But for integrable functions $\nabla \wedge \nabla = 0$. In this case we have $\nabla^2 = \nabla \cdot \nabla$.

The following form of the product rule deviates from [8] insofar as we do not use the perhaps unfamiliar overdot notation of Hestenes and Sobczyk.

Proposition 3.3 (Product rule).

$$\nabla(fg) = (\nabla f)g + \nabla_{\boldsymbol{a}} f(\boldsymbol{a} \cdot \nabla g) = (\nabla f)g + \sum_{k=1}^{n} \boldsymbol{e}_k f(\partial_k g)\,. \tag{3.5}$$

Note that the multivector functions f and g in (3.5) *do not necessarily commute.*

Proposition 3.4 (Integration by parts).

$$\int_{\mathbb{R}^n} g(\boldsymbol{x})[\boldsymbol{a} \cdot \nabla h(\boldsymbol{x})] d^n\boldsymbol{x} = \left[\int_{\mathbb{R}^{n-1}} g(\boldsymbol{x}) h(\boldsymbol{x}) d^{n-1}\boldsymbol{x}\right]_{a \cdot x = -\infty}^{a \cdot x = \infty} - \int_{\mathbb{R}^n} [\boldsymbol{a} \cdot \nabla g(\boldsymbol{x})] h(\boldsymbol{x}) d^n\boldsymbol{x}. \tag{3.6}$$

Remark 3.5. Proposition 3.4 reduces to the familiar coordinate form, if we insert for $\boldsymbol{a}$ the grade 1 basis vectors $\boldsymbol{e}_k, 1 \leq k \leq n$ of (2.1), because

$$\boldsymbol{e}_k \cdot \nabla = \partial_k \quad \text{and} \quad \boldsymbol{e}_k \cdot \boldsymbol{x} = x_k. \tag{3.7}$$

But since the introduction of a coordinate system is arbitrary, we can conversly always rotate every chosen coordinate vector into the direction of the vector $\boldsymbol{a}$ of proposition 3.4, which shows that the generalized form 3.4 for the integration by parts formula is valid. Proposition 3.4 is used in the proof of the directional uncertainty principle 5.1.

4. Clifford Fourier Transform (CFT)

Definition 4.1. The Clifford Fourier transform (CFT) of $f(\boldsymbol{x})$ is the function $\mathcal{F}\{f\}$: $\mathbb{R}^n \to \mathcal{G}_n$ given by

$$\mathcal{F}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^n} f(\boldsymbol{x})\, e^{-i_n \boldsymbol{\omega} \cdot \boldsymbol{x}}\, d^n\boldsymbol{x}, \tag{4.1}$$

where we can write $\boldsymbol{\omega} = \omega_1 \boldsymbol{e}_1 + \omega_2 \boldsymbol{e}_2 + \ldots + \omega_n \boldsymbol{e}_n$, $\boldsymbol{x} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \ldots + x_n \boldsymbol{e}_n$ with $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ the basis vectors of $\mathbb{R}^n$.

Note that

$$d^n\boldsymbol{x} = \frac{d\boldsymbol{x_1} \wedge d\boldsymbol{x_2} \wedge \ldots \wedge d\boldsymbol{x_n}}{i_n} \tag{4.2}$$

TABLE 1. Properties of the Clifford Fourier transform (CFT)

Property	**Multivector Function**	**CFT**
Linearity	$\alpha f(\boldsymbol{x})+\beta\, g(\boldsymbol{x})$	$\alpha\mathcal{F}\{f\}(\boldsymbol{\omega})+\,\beta\mathcal{F}\{g\}(\boldsymbol{\omega})$
Delay	$f(\boldsymbol{x}-\boldsymbol{a})$	$e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{a}}\mathcal{F}\{f\}(\boldsymbol{\omega})$
Shift	$e^{i_n\boldsymbol{\omega}_0\cdot\boldsymbol{x}}f(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega}-\boldsymbol{\omega}_0)$
Scaling	$f(a\boldsymbol{x}),\ a\in\mathbb{R}\setminus\{0\}$	$\frac{1}{\lvert a\rvert^n}\mathcal{F}\{f\}(\frac{\boldsymbol{\omega}}{a})$
Convolution	$(f\star g)(\boldsymbol{x})$	$\mathcal{F}\{f\}(\boldsymbol{\omega})\,\mathcal{F}\{g\}(\boldsymbol{\omega})$
Vec. diff.	$\boldsymbol{a}\cdot\nabla f(\boldsymbol{x})$	$i_n\ \boldsymbol{a}\cdot\boldsymbol{\omega}\mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{a}\cdot\boldsymbol{x}\ f(\boldsymbol{x})$	$i_n\ \boldsymbol{a}\cdot\nabla_{\boldsymbol{\omega}}\ \mathcal{F}\{f\}(\boldsymbol{\omega})$
	$\boldsymbol{x}f(\boldsymbol{x})$	$i_n\ \nabla_{\boldsymbol{\omega}}\ \mathcal{F}\{f\}(\boldsymbol{\omega})$
Vec. deriv.	$\nabla^m f(\boldsymbol{x})$	$(i_n\ \boldsymbol{\omega})^m\mathcal{F}\{f\}(\boldsymbol{\omega})$
Plancherel T.	$\int_{\mathbb{R}^n} f_1(\boldsymbol{x})\widetilde{f_2(\boldsymbol{x})}\,d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}\{f_1\}(\boldsymbol{\omega})\widetilde{\mathcal{F}\{f_2\}(\boldsymbol{\omega})}\,d^n\boldsymbol{\omega}$
sc. Parseval T.	$\int_{\mathbb{R}^n}\ \lVert f(\boldsymbol{x})\rVert^2\,d^n\boldsymbol{x}$	$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\ \lVert\mathcal{F}\{f\}(\boldsymbol{\omega})\rVert^2\,d^n\boldsymbol{\omega}$

is scalar valued ($d\boldsymbol{x_k}=dx_k\boldsymbol{e_k}$, $k=1,2,\ldots,n$, no summation). For the dimension $n=3(\mathrm{mod}\,4)$ the pseudoscalar i_n acts like a commutative[4] imaginary unit ($i_n^2=-1$), i.e. i_n commutes with every element of $\mathcal{G}_n$ (it is *central*), and hence the Clifford Fourier kernel $e^{-i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}$ will also commute with every element of $\mathcal{G}_n$. We therefore have the isomorphism (2.3) [exemplified for $n=3$ in (2.2)]. And in consequence, we also have an isomorphism between the presented Fourier transform and the classical Fourier transform, which also provides a straighforward strategy for the proofs of the properties of the CFT listed in table 1. An alternative way would be to generalize the proofs for $n=3$ in [6] to $n=3(\mathrm{mod}\,4)$. Due to the isomorphism, the CFT of equation (4.2) can be broken down to a tupel of 2^{n-1} scalar complex Fourier transforms, which also permits for numerical applications to make use of well-established fast Fourier transform algorithms. This has already been exploited for $n=3$ in [7].

Theorem 4.2. *The Clifford Fourier transform $\mathcal{F}\{f\}$ of $f\in L^2(\mathbb{R}^n,\mathcal{G}_n)$, $\int_{\mathbb{R}^n}\lVert f\rVert^2 d^n\boldsymbol{x}<\infty$ is invertible and its inverse is calculated by*

$$\mathcal{F}^{-1}[\mathcal{F}\{f\}](\boldsymbol{x})=f(\boldsymbol{x})=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\mathcal{F}\{f\}(\boldsymbol{\omega})\,e^{i_n\boldsymbol{\omega}\cdot\boldsymbol{x}}\,d^n\boldsymbol{\omega}. \tag{4.3}$$

A number of properties of the CFT are listed in table 1. A related formula for polynomials of the vector derivative (compare line 9) can be found in [4]. The reverse of line 10 and the square norm of line 11 are defined in (2.5) and (2.6), respectively.

[4]It is possible to define the CFT for $n=2(\mathrm{mod}\,4)$ as well, but then care has to be taken of the general non-commutativity of i_n with the elements of $\mathcal{G}_n$.

5. Uncertainty Principle

The uncertainty principle plays an important role in the development and understanding of quantum physics. It is also central for information processing [10].

In quantum physics it states e.g. that particle momentum and position cannot be simultaneously known. The multivector function $f(\boldsymbol{x})$ would represent the spatial part of a separable wave function and its CFT $\mathcal{F}\{f\}(\boldsymbol{\omega})$ the same wave function in momentum space (compare [11, 12, 13]). The variance in space would then be calculated as $(k = 1, 2, 3)$

$$(\Delta x_k)^2 = \int_{\mathbb{R}^3} \langle f(\boldsymbol{x})(\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \tilde{f}(\boldsymbol{x})\rangle \, d^3\boldsymbol{x} = \int_{\mathbb{R}^3} (\boldsymbol{e}_k \cdot \boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x},$$

where it is customary to set without loss of generality the mean value of $\boldsymbol{e}_k \cdot \boldsymbol{x}$ to zero [13]. The variance in momentum space would be calculated as $(l = 1, 2, 3)$

$$\begin{aligned}(\Delta\omega_l)^2 &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \langle \mathcal{F}\{f\}(\boldsymbol{\omega})(\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \widetilde{\mathcal{F}}\{f\}(\boldsymbol{\omega})\rangle \, d^3\boldsymbol{\omega} \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} (\boldsymbol{e}_l \cdot \boldsymbol{\omega})^2 \, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^3\boldsymbol{\omega}.\end{aligned}$$

Again the mean value of $\boldsymbol{e}_l \cdot \boldsymbol{\omega}$ is costumarily set to zero, it merely corresponds to a phase shift [13]. Using our mathematical units, the space-momentum uncertainty relation of quantum mechanics is then expressed by (compare e.g. with (4.9) of [12, page 86])

$$\Delta x_k \Delta\omega_l = \frac{1}{2}\,\delta_{k,l} F, \tag{5.1}$$

where $\delta_{k,l}$ is the usual Kronecker symbol. Note that we have not normalized the squares of the variances by division with $F = \int_{\mathbb{R}^3} \|f(\boldsymbol{x})\|^2 \, d^3\boldsymbol{x}$, therefore the extra factor F on the right side of (5.1). Further explicit examples from image processing can be found in [17].

In general in Fourier analysis such conjugate entities correspond to the variances of a function and its Fourier transform which cannot both be simultaneously sharply localized (e.g. [10, 14]). Material on the classical uncertainty principle for the general case of $L_2(\mathbb{R}^n)$ without the additional condition $\lim_{|x|\to\infty} |x|^2|f(x)| = 0$ can be found in [15] and [16]. Felsberg [17] even notes for two dimensions: *In 2D however, the uncertainty relation is still an open problem. In* [18] *it is stated that there is no straightforward formulation for the 2D uncertainty relation.*

From the view point of geometric algebra an uncertainty principle gives us information about how the variance of a multivector valued function and the variance of its Clifford Fourier transform are related. We can shed the restriction to the parallel $(k = l)$ and orthogonal $(k \neq l)$ cases of (5.1) by looking at the $\boldsymbol{x} \in \mathbb{R}^n$ variance in an arbitrary but fixed direction $\boldsymbol{a} \in \mathbb{R}^n$ and at the $\boldsymbol{\omega} \in \mathbb{R}^n$ variance in an arbitrary but fixed direction $\boldsymbol{b} \in \mathbb{R}^n$. This leads to the following theorem.

Theorem 5.1 (Directional uncertainty principle). *Let f be a multivector valued function in $\mathcal{G}_n$, $n = 3 \,(\mathrm{mod}\, 4)$, which has the Clifford Fourier transform $\mathcal{F}\{f\}(\boldsymbol{\omega})$.*

Assume $\int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} = F < \infty$*, then the following inequality holds for arbitrary constant vectors* $\boldsymbol{a}$*,* $\boldsymbol{b}$*:*

$$\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2 \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \geq (\boldsymbol{a}\cdot\boldsymbol{b})^2\, \frac{1}{4}\, F^2 \quad (5.2)$$

Proof. Applying the results stated so far we have[5]

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\overset{\text{table 1, line 6}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \|\mathcal{F}\{\boldsymbol{b}\cdot\nabla f\}(\boldsymbol{\omega})\|^2\, d^n\boldsymbol{\omega} \\
&\overset{\text{sc. Parseval}}{=} \int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} \int_{\mathbb{R}^n} \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x} \\
&\overset{\text{footnote 5}}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x}\, \|f(\boldsymbol{x})\|\, \|\boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\|\, d^n\boldsymbol{x}\right)^2 \\
&\overset{(2.9)}{\geq} \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} |\langle \widetilde{f(\boldsymbol{x})}\, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle|\, d^n\boldsymbol{x}\right)^2 \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x} \langle \widetilde{f(\boldsymbol{x})}\, \boldsymbol{b}\cdot\nabla f(\boldsymbol{x})\rangle\, d^n\boldsymbol{x}\right)^2.
\end{aligned}$$

Because of (2.6) and (2.7)

$$(\boldsymbol{b}\cdot\nabla)\|f\|^2 = 2\langle \widetilde{f}\, (\boldsymbol{b}\cdot\nabla) f\rangle, \quad (5.3)$$

we furthermore obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\boldsymbol{a}\cdot\boldsymbol{x})^2\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\, \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} (\boldsymbol{b}\cdot\boldsymbol{\omega})^2\, \|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega} \\
&\geq \left(\int_{\mathbb{R}^n} \boldsymbol{a}\cdot\boldsymbol{x}\, \frac{1}{2}\, (\boldsymbol{b}\cdot\nabla\|f\|^2)\, d^n\boldsymbol{x}\right)^2 \\
&\overset{\text{Prop. 3.4}}{=} \frac{1}{4}\left(\left[\int_{\mathbb{R}^{n-1}} \boldsymbol{a}\cdot\boldsymbol{x} \|f(\boldsymbol{x})\|^2 d^{n-1}\boldsymbol{x}\right]_{b\cdot x=-\infty}^{b\cdot x=\infty} \right. \\
&\qquad \left. - \int_{\mathbb{R}^n} [(\boldsymbol{b}\cdot\nabla)(\boldsymbol{a}\cdot\boldsymbol{x})]\, \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x}\right)^2 \\
&= \frac{1}{4}\left(0 - \boldsymbol{a}\cdot\boldsymbol{b}\int_{\mathbb{R}^n} \|f(\boldsymbol{x})\|^2\, d^n\boldsymbol{x})\right)^2 \\
&= (\boldsymbol{a}\cdot\boldsymbol{b})^2\, \frac{1}{4}\, F^2.
\end{aligned}$$

Choosing $\boldsymbol{b} = \pm\boldsymbol{a}$, i.e. $\boldsymbol{b} \parallel \boldsymbol{a}$, with $\boldsymbol{a}^2 = 1$ we get from theorem 5.1 the **uncertainty principle** for parallel variance directions [compare with case $k = l$ of (5.1)]:

[5] $\phi, \psi : \mathbb{R}^n \to \mathbb{C}$, $\quad \int_{\mathbb{R}^n} |\phi(x)|^2 d^n x \int_{\mathbb{R}^n} |\psi(x)|^2 d^n x \geq (\int_{\mathbb{R}^n} \phi(x)\bar{\psi(x)}\, d^n x)^2$

$$\int_{\mathbb{R}^n}(\boldsymbol{a}\cdot\boldsymbol{x})^2\|f(\boldsymbol{x})\|^2\,d^n\boldsymbol{x}\,\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\boldsymbol{a}\cdot\boldsymbol{\omega})^2\,\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\,\geq\,\frac{1}{4}\,F^2. \tag{5.4}$$

□

Remark 5.2. In (5.4) equality holds for *Gaussian* multivector valued functions

$$f(\boldsymbol{x}) = C_0\, e^{-k\,\boldsymbol{x}^2} \tag{5.5}$$

where $C_0 \in \mathcal{G}_n$ is an arbitrary but constant multivector, $0 < k \in \mathbb{R}$. The proof for this follows from the observation that we have for the f of (5.5)

$$-2k\,\boldsymbol{a}\cdot\boldsymbol{x}\,f = \boldsymbol{a}\cdot\nabla f. \tag{5.6}$$

Choosing orthogonal directions $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^n$ we get from theorem 5.1 the **uncertainty principle** for orthogonal variance directions [compare with case $k \neq l$ of (5.1)]:

Theorem 5.3. *For* $\boldsymbol{a}\cdot\boldsymbol{b}$ *=0, i.e.* $\boldsymbol{b}\perp\boldsymbol{a}$*, we get*

$$\int_{\mathbb{R}^n}(\boldsymbol{a}\cdot\boldsymbol{x})^2\|f(\boldsymbol{x})\|^2\,d^n\boldsymbol{x}\,\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\boldsymbol{b}\cdot\boldsymbol{\omega})^2\,\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\,\geq\,0. \tag{5.7}$$

Theorem 5.4. *Under the same assumptions as in theorem* 5.1*, we obtain*

$$\int_{\mathbb{R}^n}\boldsymbol{x^2}\,\|f(\boldsymbol{x})\|^2\,d^n\boldsymbol{x}\,\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\boldsymbol{\omega^2}\,\|\mathcal{F}\{f\}(\boldsymbol{\omega})\|^2 d^n\boldsymbol{\omega}\,\geq\,\frac{n}{4}\,F^2. \tag{5.8}$$

Remark 5.5. For the proof of theorem 5.4 we first insert $\boldsymbol{x}^2 = \sum_{k=1}^n(\boldsymbol{e}_k\cdot\boldsymbol{x})^2, \boldsymbol{\omega}^2 = \sum_{l=1}^n(\boldsymbol{e}_l\cdot\boldsymbol{\omega})^2$. After that we apply (5.4) and (5.7) depending on the relative directions of the vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}_l$.

6. Conclusions

The (real) Clifford Fourier transform extends the traditional Fourier transform on scalar functions to $\mathcal{G}_n$ multivector functions with $n = 3\,(\mathrm{mod}\,4)$. Basic properties and rules for differentiation, convolution, the Plancherel and Parseval theorems were introduced.[6] We then presented a general directional uncertainty principle in the geometric algebra $\mathcal{G}_n$, $n = 3\,(\mathrm{mod}\,4)$ which describes how the variances (in arbitrary but fixed directions) of a multivector-valued function and its Clifford Fourier transform relate. The formula of the uncertainty principle in $\mathcal{G}_n$, $n = 3\,(\mathrm{mod}\,4)$ can be extended to $\mathcal{G}_n$, $n = 2\,(\mathrm{mod}\,4)$ taking due care of the resulting general non-commutativity of i_n with the elements of $\mathcal{G}_n$.

It is known that the Fourier transform is successfully applied to solving equations in all of classical and quantum physics such as the heat equation, wave equations, etc. The same is true for applications of the Fourier transform to problems in image processing and signal theory. Therefore in the future, we can apply geometric algebra and the Clifford Fourier transform to solve such problems involving the whole range of k-vector fields ($k = 0, 1, 2, \ldots, n$) in geometric algebras $\mathcal{G}_n$ with $n = 3\,(\mathrm{mod}\,4)$ and study the inevitably remaining uncertainties of the solutions.

[6] Similar formulas for $n = 2$ are also given and applied in [7].

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